Pricing an Option on a Non-Decreasing Asset Value: An Application to Movie Revenue

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Abstract

Modeling the revenue function of a new innovation has been addressed by a number of researchers. The marketing literature focuses on the introduction and acceptance of new products. Likewise, the adoption pattern and eventual economic success of a newly released movie can be viewed as an innovation subject to many of the same forces. Recently there have been references in the popular literature to option writing on the revenue of newly released films. Unique technical problems to be addressed include 1) the requirement that the total revenue function be non-decreasing over time and 2) the lack of observations on the innovation at the time of release. In addition, the great majority of extant innovation models are deterministic in nature, quantifying only the expected number of adopters or expected revenue function. Option pricing, on the other hand, is driven in large part by the volatility of the underlying process. In this paper we develop general properties and derive formulas for options on movie revenues. Parameter estimates and simulations are also provided.

Classification words: options, film finance, financial engineering, derivatives on non-traded assets, innovation models

PRICING AN OPTION ON A NON-DECREASING ASSET VALUE: AN APPLICATION TO MOVIE REVENUE

Over the last twenty years, financial engineers have created an impressive array of instruments designed to manage the risk of uncertain market values and cash flows. In recent years, financial engineering has headed to Hollywood – so to speak – by creating instruments designed to manage the financial risk associated with motion pictures. These instruments take the form of securitized, equity-like claims on motion picture revenues as well as options on those revenues. Options on movie revenues are unique in several ways and pose many interesting and challenging problems in volatility modeling and pricing.

For example, these claims are typically sold before any revenue is generated. Thus, the value of the underlying asset is technically unknown. Typically, option pricing models require a current value for the underlying. Here the underlying has not been publicly traded, rendering the underlying much like that of an initial public offering. In many cases, however, the underlying asset will never be traded, although the claimant on the underlying asset will receive cash flows. In some cases, however, the instrument is securitized, so there would presumably be a market price at which it would sell, though it is quite possible that the market would be fairly illiquid.

A second complication is that the underlying is non-decreasing. Although revenue can theoretically remain constant over a time period, it would typically increase. These characteristics are most unusual compared to the underlyings of options typically observed in financial markets. Diffusion models would not meet the necessary requirements. Moreover, not only does revenue only increase but it does so in finite amounts. Hence, even if smooth diffusion models could be used, they would be unlikely to exhibit the appropriate properties.

Another interesting problem is that movies normally exhibit an extremely high degree of uncertainty at the start. After the first few weeks have passed and the critic reviews have been published, most of the uncertainty has been resolved. Of course, the options can be designed to expire either quickly or much later.

Although standard European options are usually the first instruments offered for new financially engineered solutions, American options could be of interest to some investors. As is well known, American calls on most assets are exercisable early only if the asset has a cash flow. We will show that this type of call option, with no specific cash flow, is optimally exercised early under some conditions. For general options, American puts are nearly always optimally exercised early under the condition of a sufficiently low value of the underlying. In this paper, we will show, however, that American puts will either be exercised the instant they are eligible for exercise or they will effectively terminate.[1] This means that American puts that are exercisable immediately will either have zero value or will be exercised at once. For that reason, such options would be uninteresting. Alternatively, Bermuda-style American options, not exercisable until

[1]By "effectively terminate", we mean that they will remain alive but have zero value because of no possibility of expiring in-the-money.

a later date, will be considered. We will show that these options are simple to price: they will either have zero value at the first exercise date or will be exercised at that time. Hence, their actual expirations are irrelevant.

The paper is organized as follows. Section 1 provides background information on options on movie revenues. Section 2 examines boundary conditions for options on movie revenues. Section 3 introduces a deterministic model of adoption that forms the basis for the stochastic model developed in Section 4. Section 5 develops pricing formulae for the valuation of options and Section 6 is a simulation of option prices. Section 7 presents the empirical estimates of model parameters and Section 8 provides conclusions.

1. SOME BACKGROUND INFORMATION ON OPTIONS ON MOVIE REVENUE

A movie is an excellent textbook example of a capital investment decision. A studio commits a significant amount of initial funding, acquires the resources (including contracts with the actors, director, and other personnel), produces the movie, and then expends additional resources in promotion and distribution while the movie generates cash flows. The cash flows from box office receipts have an exceptionally short life. For example, one of the most successful movies of all time, Titanic, lasted in theaters only nine months. In contrast one of the least successful, Gigli, lasted only about two weeks. Of course, most movies also have a second life, generating cash flows through video rentals, video sales, and television rights. These amounts can be substantial. For example, Titanic was released on video in 1999 and has since generated another $300 million from rentals[2]. A third major source of revenue from a movie is box office receipts, video sales, and rentals in non U. S. countries. Titanic, for example, has generated about twice as much in box office revenue and almost three times as much in video rentals outside the U. S. as inside the U. S. Foreign cash flows can occur almost simultaneously with U. S. cash flows, but we do not consider these revenues in our analysis nor do we consider within country re-released movie income.[3]

As a risky investment, a movie is characterized by a tremendous concentration of uncertainty during first few weeks after release. A number of factors determine success or failure of a movie but the process is extremely complex. Expensive movies with well-known stars are sometimes dismal failures, while low-budget movies are sometimes highly successful.[4] Some movies receive excellent critical reviews and awards but are commercially disappointing.[5] It is extremely common to observe movies that do well in spite of poor critic reviews. Movies that are commercial failures in the U. S. are oftentimes highly successful in foreign markets.[6] Because of the high

[2]We ignore video and television revenues in our model.

[3]For example, *Close Encounters of the Third Kind* was released in 1977 and again in 1980 with additional footage. *Star Wars* was first released in 1977 and has been re-leased five times in the U.S.

[4]For example, *The Alamo* released in 2004 starring Billy Bob Thornton and Dennis Quaid cost $95 million and earned about $22 million in box office revenues. At the other extreme, *The Blair Witch Project* released in 1999 cost about $35,000 and generated box office receipts of over $140 million.

[5]For example, *Bravehart* generated only abouyt $76 million in revenue in the U.S., just slightly above its cost of $72 million. The movie received 10 Oscar nominationsand won five, including Best Picture.

[6]For example, in 2004 the movie *Troy*, costing $185 million, earned only $133 million

degree of uncertainty and the large financial investment, movies are prime candidates for the financial engineering of risk transfer instruments.

One of the first such instruments was a $400 million seven-year Eurobond released in 1992 by The Walt Disney Company. The interest rate was tied to the revenues from a combination of 13 Disney movies released in Europe. The rate was set at $7\,\frac{1}{2}\%$ for 18 months. Beyond that point, the coupon was set at a formula directly related to revenues from these movies. The total rate would end up being between 3% and 13.5%. Obviously the return on this bond takes on the characteristic of a call option, exercising and paying additional money if target revenue levels are met.

In 1997, Risk magazine (Conway, 1997) reported on the creation of an Entertainment Industry Options Exchange in London. This exchange was conceived as an organized marketplace for trading derivatives based on movies and other entertainment-based revenues. One of the first instruments planned was options based on the album *Perfect World* by pop singer Debbie Bonham, sister of the late Led Zeppelin drummer John Bonham. There is no subsequent evidence that this exchange was ever formally operational.

Also in 1997 pop singer David Bowie released $55 million of bonds with coupons tied to the revenue from some of his albums. These instruments became known as Bowie Bonds, but they were not successful for investors, however, and were downgraded to near junk status in 2004.

in U.S. box office receipts but has grossed about $350 million outside the U.S., placing it in the top 50 movies of all time.

In 2004 Risk (Patel, 2004) describes the efforts of an American company called Center-Group to create an electronic market for new derivative instruments based on box office receipts. No further evidence exists of whether this market has been formally created. The article also references the securitization of revenues of such movie studios such as Vivendi and Dreamworks SKG. It also notes, however, that some defaults have occurred in previous securitizations.

Virtual derivatives on movies can be traded on the Hollywood Stock Exchange (www.hsx.com), a subsidiary of the American bond trading firm Cantor Fitzgerald. This exchange was created in 1996, and all trading is based on fictional (virtual) money. Stocks and options on movies can be purchased and sold. The exchange also offers "bonds" on actors and actresses in which value is accrued based on revenues generated by their movies. The exchange creates value by selling the data to the movie industry, where it could presumably be used to assess the public's interest in movies and performers.

While it is not clear that derivatives on revenues from movies, music recordings, and tours have been widespread, there would appear to be great potential for such instruments. Americans spend more than $13 billion a year on movies. Not only is the entertainment industry in need of risk management techniques, but claims on these instruments could be particularly attractive to investors because of diversification potential. Hedge funds and other institutional investors would seem to be an ideal market, but individual investors might well become interested, as evidenced by the over one million participants as claimed by the Hollywood Stock Exchange.

Although numerous types of derivatives are potentially viable for entertainment-based products, our focus will be on options on movie revenue. As briefly mentioned in the introduction, these options have several unique features that must be addressed in building pricing models. These features pose interesting and unusual challenges for pricing these types of claims. An added benefit of this research is that modeling the revenue stream would be beneficial for constructing securitized equity shares. We confine our research to European and American options. Exotic variations will undoubtedly be quite interesting, but we leave that subject to future research.

2. BOUNDARY CONDITIONS FOR OPTIONS ON MOVIE REVENUES

We begin by examining the basic pricing results that can be developed without specifying a stochastic process for movie revenues. We start by defining the underlying revenue stream as a random value $R(0, s)$, which represents revenue accrued over the period 0 to s. Although a discrete-time model could be used here, we let revenue accrue continuously. The instantaneous revenue at time t is denoted as $R(t)$. Hence,

$$R(0, s) = \int_0^s dR. \tag{1}$$

By definition, $R(0, 0) = R(0) = 0$. The revenue stream always increases; therefore, $R(0, s) > R(0, t)$ for any $s > t$.[7] Initially we consider only

[7]One interesting characteristic of this type of underlying is that even though it is

European options on this revenue stream with exercise price K. These options expire at time T. The continuously compounded risk-free rate is r. The current values of the options are C and P. The values at expiration are $C(T)$ and $P(T)$.

As noted earlier, barring securitization, the underlying revenue stream is not a traded asset. Nonetheless, the revenue stream has economic value, because the holder of the stream has a claim on the expected revenue, $R(0,T)$. Assuming a discount rate of k to reflect the risk-free rate plus a risk premium, we define the value of the revenue stream at time 0 as

$$V(0) = E[R(0,T)]e^{-kT}. \tag{2}$$

If $s > 0$, a portion of the accrued revenue is known. Splitting the revenue stream into the known and unknown components, we have

$$R(0,T) = \int_0^s dR + \int_s^T dR, \tag{3}$$

and the valuation formula at s for total revenue at T becomes

$$V(s) = R(0,s)e^{-r(T-s)} + e^{-k(T-s)}E\left[\int_s^T dR\right]. \tag{4}$$

Conditioned on known values at time s, the value is divided into riskless and risky streams. Let the discounted integral expression be written as $\Omega(s,T)$, which represents the present value of the remaining unknown cash

money, there is no adjustment for the time value of money. That is, the option is on the cumulative revenue stream not adjusted for interest. While such an adjustment would be simple, we do not do it here, in keeping with the typical structure as suggested by instruments that have been created or discussed.

flows. Then the value of the claim is

$$V(s) = R(0,s)e^{-r(T-s)} + \Omega(s,T). \tag{5}$$

As s approaches T, $\Omega(s,T)$ becomes smaller and the value of the revenue stream approaches the value of a riskless bond accruing interest. In the remainder of this section, we assume that the value $V(t)$ can be obtained.

2.1. Option Payoff at Expiration

By construction, the payoffs of the options at expiration are

$$\begin{aligned} C(T) &= Max\{0, R(0,T) - K\}) \quad (6) \\ P(T) &= Max\{0, K - R(0,T)\} \end{aligned}$$

Until indicated, all options are European.

2.2. Maximum Values

Now consider the maximum possible values of these options prior to expiration. Without loss of generality, position ourselves at time 0. Since there is no upper limit to the revenue stream, the maximum value of the call is infinite. Theoretically, a movie could generate no revenue; hence, the maximum value of a put would be the present value of K:

$$\begin{aligned} C &\leq \infty \quad (7) \\ P &\leq Ke^{-rT} \end{aligned}$$

In practice, there may well be an extremely large upper limit to the revenue because there is surely a maximum amount of revenue that would be generated if every person attended a movie. While this maximum will play a role in our pricing models, we do not invoke its effect here.

2.3. Minimum Values

Now consider the minimum values, sometimes known as the lower bounds. As previously developed, the holder of the revenue stream has a claim worth V. Suppose that the holder borrows the present value of K and sells the call. The payoffs are $R(0,T) - K - (R(0,T) - K) = 0$ if $R(0,T) > K$ and $R(0,T) - K \leq 0$ otherwise. Thus, this combination has a non-positive payoff and, therefore, must have a non-positive current value. Hence, $V - Ke^{-rT} - C \leq 0$ and thus $C \geq V - Ke^{-rT}$. Given the breakdown of V, we could write this as $C \geq R(0,s)e^{-rT} + \Omega(s,T) - Ke^{-rT}$.

Now consider the put. Let the holder of the revenue claim buy a put and borrow the present value of K. The payoffs are $R(0,T) + (K - R(0,T)) - K = 0$ if $R(0,T) < K$ and $R(0,T) - K \geq 0$ otherwise. Since these payoff values are non-negative, the current value must be non-negative. Hence, $V + P - Ke^{-rT} \geq 0$ and $P \geq Ke^{-rT} - V$. Thus, the lower bounds are

$$\begin{aligned} C &\geq Max\{0, V - Ke^{-rT}\}, \qquad (8) \\ P &\geq Max\{0, Ke^{-rT} - V\}. \end{aligned}$$

Of course, these are the same as the lower bounds for ordinary options.

2.4. Put-Call Parity

Put-call parity easily follows. The payoffs of a portfolio combining the revenue stream and a put are $R(0,T) + K - R(0,T) = K$ if $R(0,T) < K$

and $R(0,T)$ otherwise. The payoffs of a portfolio combining a risk-free bond and a call are K if $R(0,T) < K$ and $K + R(0,T) - K = R(0,T)$ otherwise. Hence, the revenue claim plus put must have the same initial value as the bond plus call. Thus,

$$V + P = C + Ke^{-rT}. \tag{9}$$

Of course, this is precisely the same form of put-call parity for standard options.

2.5. The Effect of Time to Expiration

Consider options with different expiration dates, T_1 and T_2 where $T_2 > T_1$. A call option is clearly worth no less with longer time to expiration, because the underlying cannot be any lower at T_2 than at T_1. A put option, however, is worth no less with the shorter expiration. The longer term penalizes a put on two counts. For one, the underlying will always be at least as high with the longer expiration. In addition, the holder of the put would forgo interest on the exercise price by waiting the longer time to exercise. Hence,

$$\begin{aligned} C_{T_2} &\geq C_{T_1}, \\ P_{T_2} &\geq P_{T_2}. \end{aligned} \tag{10}$$

Note how the put result is slightly different than the result for standard puts where additional time is beneficial but must be traded off against lost interest on the exercise price. For a put on revenue, however, the additional time conveys no benefits, because revenue can only increase.

2.6. The Effect of Exercise Price

Now consider the effect of exercise price. Consider two options with different exercise prices, K_1 and K_2 with $K_2 > K_1$. A call with a lower exercise price must be worth no less than a call with a higher exercise price. A put with a higher exercise price must be worth no less than a put with a lower exercise price so

$$\begin{aligned} C_{K_1} &\geq C_{K_2}, \qquad (11) \\ P_{K_2} &\geq P_{K_1}. \end{aligned}$$

Obviously these are the same results as for standard options.

2.7. American Options

Consider the properties of American options. For call options, we assume unrestricted exercise at any time prior to expiration. For put options, however, early exercise must be Bermuda-style, that is, prohibited up to a certain point. Otherwise, an American put would be exercised immediately, because any additional time can only make the put move less in-the-money or deeper out-of-the-money. Hence, we assume that the put cannot be exercised until time τ.

The American call will have the same expiration payoff as the European call. The upper limit for the European call of infinity can be no lower for the American call, because the latter can always be treated like a European call and not exercised early. Since at any time $s < T$ the American call can be exercised for $R(0, s) - K$, we must consider whether early exercise is ever supported. Recall that the lower bound is $V(s) - Ke^{-r(T-s)}$. A sufficient condition for no early exercise is that $V(s) - Ke^{-r(T-s)} > R(0,t) - K$. Rearranging, we obtain

$$(R(0,s) - K)(1 - e^{-r(T-s)}) < \Omega(s,T) \tag{12}$$

The left-hand side is the interest that would be earned on capturing the exercise value, and the right-hand side is the expected gain from the remaining unknown revenue. Since this condition will not always be met, the American call can be exercised early. Without an option pricing model we cannot identify precisely when this would occur. But we know that the option would sell for nearly its minimum value when the uncertainty is low. Let us assume that the value of the remaining revenue, $\Omega(s,T)$, represents a small and virtually risk-free stream. Then, the call would sell for approximately its minimum value, $R(0,s)e^{-r(T-s)} + \Omega(s,T) - Ke^{-r(T-s)}$. This value is less than the exercise value, $R(0,s) - K$ if $(R(0,s) - K)(1 - e^{-r(T-s)}) > \Omega(s,T)$. The left-hand side represents interest on the exercise value and the right-hand side represents the remaining uncertainty. This result suggests that a deep in-the-money option will almost surely be exercised early so that the exercise value can be claimed immediately and reinvested to earn more than the value expected from holding the position. In fact, exercise might even occur quite early, because uncertainty is resolved quickly and the interest that could be earned would be largest early in the life of the option.

Given the possibility of early exercise of the call, its lower bound is, therefore, $Max\{0, V - K, V - Ke^{-rT}\}$and $C^A > C$. These characteristics of American calls are most unusual compared to standard American calls, which are never exercised early unless there is a cash flow on the underlying. In this case, however, the underlying is nothing but a sequence of cash flows, justifying early exercise in some cases.

Now consider Bermuda-style American puts. Move forward to time τ, the first point at which the put can be exercised. Suppose during the entire period up to τ, the revenue did not exceed the strike. Then at τ, it would have to be the case that $V(\tau) < K$, so the option would be immediately exercised for a value of $K - V(\tau)$. If at any time prior to τ, revenue exceeded the strike, then the option would never be able to move in-the-money, so its value is effectively zero. These results show that the American put on revenue is a simple instrument. In fact, we can not only characterize the bounds of this instrument: we can nearly obtain a pricing formula. For all states in which revenue exceeds K prior to τ, revenue must also exceed K at τ. In such states, the option is worthless at τ, even though it technically is still alive. The probability of this occurring is $Prob(R(0, \tau) > K)$. For all complementary states, where revenue does not exceed K prior to τ, the option will be exercised and pay $K - V(\tau)$ at τ. Hence, this portion of the option payoff can be represented as $(1 - Prob(R(0, \tau) > K))E(K - R(0, \tau)|R(0, \tau) < K)$. Discounting this value to the present gives the American put price.

Thus, we see that the contractual expiration of the American put is meaningless. Only τ, the first time it can be exercised, is relevant. Thus, an American put expiring at T, not exercisable until τ, is effectively a European put expiring at τ. Hence, the maximum put value prior to expiration will be $Ke^{-r\tau}$. The minimum value can be found by constructing the same portfolio as for the European put but changing the expiration to τ. The lowest price is therefore the lowest price of the European put with expiration τ.

Another variation of an American put would be one with a deterministically increasing exercise price. Let the exercise price be set at K time 0. At time s, the exercise price is $K(s)$, where $K(s)$ is any reasonable deterministic formula. We know that at time t the put must be worth at least its European value of $K(T)e^{-r(T-s)} - V(s)$. A sufficient condition for no early exercise is that $K(T)e^{-r(T-s)} - V(s) > K(s) - R(s)$. Substituting for $V(s)$, we have

$$K(T)e^{-r(T-s)} - K(s) - R(0,s)(1 - e^{-r(T-s)}) > \Omega(s,T). \qquad (13)$$

The left-hand side is the interest earned on the exercise value, taking into account the difference in strikes at times s and T. The right-hand side is the present value of the expected remaining revenue.

In the special case where the exercise price grows at the continuous risk-free rate, then $K(s) = K(T)e^{-r(T-s)}$ and we have the simple condition $R(0,s)(1 - e^{-r(T-s)}) < \Omega(s,T)$, meaning that early exercise will not occur if the interest on the revenue accrued is less than the present value of the expected remaining revenue.

A full pricing model will require modeling the stochastic process of revenue growth. To establish a framework for a stochastic model, we begin with a deterministic model that provides valuable insights on the manner in which consumers might choose to see a movie.

3. A DETERMINISTIC MODEL OF THE ADOPTION OF AN INNOVATION

Modeling the process by which movie revenue is generated has been the subject of a number of papers in the economics and marketing literature. Most of these models focus on predicting revenues after most of the uncertainty is resolved. That is, most of the models use such information as box office receipts in the first week or two and Oscar nominations to explain future revenue. The model of Sawhney and Eliashberg (1996) is primarily a time-series model, which is based on the notion that the time it takes a consumer to see a movie is the sum of the time it takes the consumer to decide to see the movie and the time it takes the consumer to act on that decision. Although the model can accommodate other information, such as number of screens, advertising, etc., the authors prefer the simplest version and apply it with what they consider good success to a sample of movies from 1990 and 1991.

Ravid (1999) develops a cross-sectional model for predicting revenue based on star power, rating, release date, number and quality of reviews, and several other measures, including information that would not be available before the movie is released. The model is tested on a sample of about 180 films released in the period 1991-1993. The purpose of the model is not to predict revenue but to explain the effect of star power in movie profitability. Nonetheless, the variables could be useful in modeling moving revenue ex ante.

Simonoff and Sparrow (2000) build a model using star power, characteristics of the movie, number of screens, production costs and a number of other variables, some of which would not be known before the movie is released. They test the model on a sample of 311 films released in 1988.

Elberse and Eliashberg (2003) use information from the domestic performance of a film to predict foreign revenues. Clearly in this case, there is a considerable amount of information available, so much of the uncertainty of our interest would already be resolved. Nonetheless, the variables they use could be helpful ex ante.

Goetzmann, Pons-Sanz, and Ravid (2004) build a model for explaining the price paid for movie scripts and the role of script prices in predicting the financial success of a movie.

For the most part, these models focus on forecasting revenue as a function of characteristics of the movie and certain time series properties. For the purposes of pricing options on movie revenue, we require a tractable and simple model for the evolution of the revenue series. Such a model should contain a drift and a volatility. The models in the literature may be useful for estimating the drift and volatility, but we must start at a more fundamental level. Moreover, we must be able to model revenue before the release of the movie.

3.1. The Revenue Function for an Innovation

Modeling the revenue function of a new innovation has been addressed by a number of researchers. The marketing literature focuses on the introduction and acceptance of new products. Likewise, the adoption pattern and eventual economic success of a newly released movie can be viewed as an innovation subject to many of the same forces. Unique technical problems to be addressed include 1) the requirement that the total revenue function be non-decreasing over time and 2) the lack of observations on the innovation at the time of release. In addition, the great majority of extant innovation models are deterministic in nature, quantifying only the expected number of adopters or expected revenue function. Option pricing, on the other hand, is driven in large part by the volatility of the underlying process.

Bass (1969), assumes that there are two forces influencing the adoption of an innovation. One force is independent of the previous number of adopters and the other force is positively influenced by the previous number of adopters. Consider first an individual adopter. The probability of event (adoption) in the interval t, $t + dt$, given that the event has not occurred previously, is called the hazard function, $h(t)$. The Bass hazard model is

$$h(t) = \frac{f(t)}{1 - F(t)} = p + qF(t), \tag{14}$$

where $f(t)$ is the density function of the time of adoption and $F(t)$ is the cumulative density up to time t. The structure of the model is determined by p and q. The parameter q must be non-negative and p must be positive. Both parameters must be finite if the density function is to be non-degenerate.

The first force in the adoption process, p, has been called the coefficient of innovation. It is the decision to adopt independent of the actions of others. Bass calls the second force, q, the coefficient of imitation. This coefficient is related to the cumulative probability of adoption up until time t. Lekvall and Wahlbin (1973) have referred to these coefficients, respectively, as the external and internal influence in the adoption process. The initial condition is $F(0) = 0$, giving the solution to equation (14) as

$$F(t) = \frac{1 - e^{-(p+q)t)}}{1 + \frac{q}{p}e^{-(p+q)t}}. \tag{15}$$

The density function for the time of adoption is then

$$f(t) = \frac{d\left(\frac{1-e^{-(p+q)t)}}{1+\frac{q}{p}e^{-(p+q)t}}\right)}{dt} = \frac{(p+q)^2\, pe^{-(p+q)t}}{\left(p + qe^{-(p+q)t}\right)^2},\ 0 < t < \infty \tag{16}$$

The density is maximum at $t^* = \frac{1}{p+q}\ln(\frac{q}{p})$. Thus , the time of maximum intensity is positive only if $q > p$. Small t^* requires $p + q$ large and $p \approx q$.

If the potential population is m, fixed, the total number adoptions up to time t under the deterministic model is $n(t) = mF(t)$ and the rate of adoptions is $\mu(0, t) \equiv mf(t)$. The same solution can be obtained by defining $F(t) = \frac{n(t)}{m}$ as the fraction of individuals who have adopted by time t. The differential equation is then written as

$$\frac{dn(t)}{dt} = (m - n(t))\left(p + \frac{q}{m}n(t)\right), \tag{17}$$

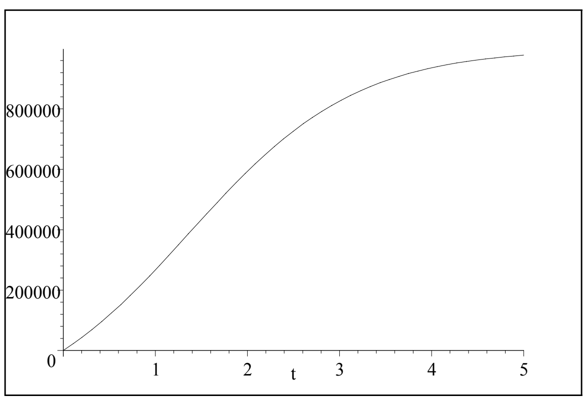
$$n(t) = m\left(\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}\right) = mF(t). \tag{18}$$

From equation (17) it is clear that the rate of adoptions is proportional to the remaining non-adopters. The coefficient of proportionality is $p+\frac{q}{m}n(t)$. Plots of total adoptions and rate of adoptions for a typical case are given below:

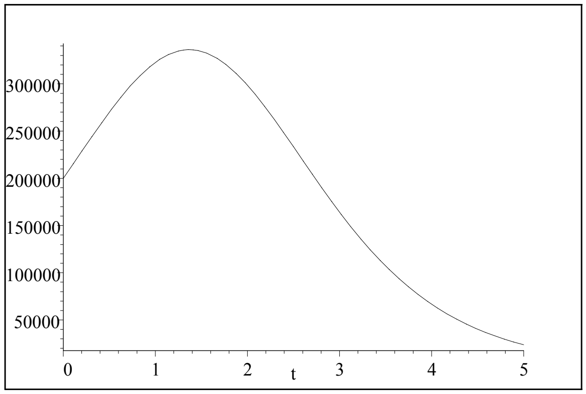
3.2. Plot of Adoptions

Let $p = 0.2$, $q = 0.9$ and $m = 1,000,000$.corresponding to adoptions that becomes asymptotic after approximately five years.

Number of Adopters (mF(t)) versus years



Rate of Adoption (mf(t)) at t-years



The number of adopters follows an s-shaped curve and intensity peaks at $t = \frac{1}{p+q}\ln(\frac{p}{q})$. Large relative values of p leads to an earlier maximum. In some cases, the peak intensity for newly released movies is near zero (just after release).

4. A STOCHASTIC REVENUE MODEL

Total revenue is assumed to be generated by a non-decreasing jump process. The process is in the tradition of the Merton-Bates jump-diffusion but without the diffusion and with non-homogeneous Poisson compounding. In the following sections, we model the number of adopters and then scale the final result by admission price per adopter. Specifically, let local change in the number of adopters be given by

$$dn = (mf(t) - \bar{y}\pi)dt + ydQ,\ mf(t) > \bar{y}\pi(t), \tag{19}$$

where $prob(dQ = 1) = \pi(t)dt + o(dt)$, $\pi(t)$ is a Poisson intensity, and y is a non-negative random variable, independent of dQ. For generality, we do not specify the distribution of y at this point. The mean of ydQ is $\bar{y}\pi dt$ so the Bass intensity $mf(t)$ is preserved in the drift of the process, i.e., locally $E(dn) = mf(t)dt$. This means that the expected value of the stochastic revenue model is consistent with the Bass deterministic model. Instantaneous drift is also non-negative when $mf(t) > \bar{y}\pi(t)$ and this condition is easily satisfied for plausible parameters.

Notice that the term ydQ determines the volatility to the process. The Poisson intensity, π, determines the frequency of jumps and the random variable y corresponds to their magnitude. Jump magnitude in the context of movies represents the jump in attendance due to the unexpected addition of new screens or other unexpected phenomena (e.g., publicity).

The solution to equation (19) is

$$n(T) = n(s) + \int_s^T (mf(t) - \bar{y}\pi)dt + \sum_{i=0}^{x(s,T)} y_i, \quad y_0 \equiv 0, \tag{20}$$

where $x(s, T)$ is the random number of jump events in the interval (s, T).

5. OPTION VALUE

European options are issued by the firm on cumulative revenue and are cash-settled at expiration. We assume a CAPM world with constant investment opportunity set and with revenue jumps uncorrelated with the market. Thus, the jump risk is diversifiable and not priced. An European call option thus has value

$$C(s,T) = ae^{-(T-s)r} E\left(Max\{0, n(s) + \int_s^T (mf(t) - \bar{y}\pi)dt + u(x) - K\}\right), \tag{21}$$

where $C(s,T)$ is the price at time-s of an option expiring at time-T, a is revenue per admission and $u(x) \equiv \sum_{i=0}^{x(s,T)} y_i$ total admission due to jumps. The strike price in currency units is aK. Using equations (20) and (21) gives by iterated expectations

$$C(t,T) = ae^{-r(T-t)} E_x E_{u|x}\left(Max\{0, n(s) + \int_s^T (mf(t) - \bar{y}\pi)dt + u(x) - K\}\right), \tag{22}$$

by iterated expectations. Therefore

$$\begin{aligned} C(t,T) &= ae^{-r(T-t)} E_x\left(\int_0^\infty Max\{0, n(s) + \mu(s,T) - \bar{y}\Phi(s,T) + u(x) - K\} f_{u|x}(u)du \right) \\ &= e^{-r(T-t)} E_x(C(x)). \end{aligned} \tag{23}$$

where $\Phi(s,T) = \int_s^T \pi(t)dt$ and $e^{-r(T-t)}C(x)$ is option value given x, where

$$C(x) = a\int_0^\infty Max\{0, n(s) + \mu(s,T) - \bar{y}\Phi(s,T) + u(x) - K\} f_{u|x}(u)du. \tag{24}$$

Let $f_{u|x=0}(u) = \delta(u)$ be the Dirac delta function for $x = 0$, otherwise f is an arbitrary pdf for $x = 1, 2, ...$[8] Define $d \equiv Max\{0,\ K + \bar{y}\Phi(s,T) -$

[8] Dirac delta, $\delta(u - u_0)$, has the property that $\delta(u - u_0) = 0$, $u \neq u_0$ and $\int_a^b \delta(u - u_0)g(u)du = g(u_0)$, $a \leq u_0 \leq b$, 0 elsewhere.

$n(s) - \mu(s,T)\}$ so that

$$C(x) = a\int_d^\infty (n(s) + \mu(s,T) - \bar{y}\Phi(s,T) + u(x) - K)\, f_{u|x}(u)du \quad (25)$$

$$= a\int_d^\infty (n + \mu + (u - \bar{y}\Phi) - K)\, f_{u|x}(u)du.$$

Notice that a sufficient condition for non-negative $Max\{\cdot\}$ integrand in equation (24) is $n+\mu > K+\bar{y}\Phi$. In this case $d = 0$ and $u \geq 0$ is the feasible sample space. If $n + \mu < K + \bar{y}\Phi$, $u \geq d = K + \bar{y}\Phi - n - \mu$ is required for a non-negative $Max\{\cdot\}$ integrand. Taking expectations over the number of jumps gives the general result

$$C(s,T) = ae^{-r(T-s)}\sum_{x=0}^{\infty} e^{-\Phi}\frac{\Phi^x}{x!}\left(\int_d^\infty (n + \mu + (u - \bar{y}\Phi) - K)\, f_{u|x}(u)du.\right) \quad (26)$$

The boundary conditions are easily verified since $s = T \Rightarrow x = 0$, $\Phi = 0$, $\mu = 0$, $u = 0$ and $d = max\{0, K - n(T)\}$. Because $f_{u|x=0}(u) = \delta(u)$ the option has value at expiration given by $C(T,T) = a\int_d^\infty (n(T) - K)\,\delta(u)du$. But $n(T) \geq K \Rightarrow \int_0^\infty (n(T) - K)\,\delta(u)du = n(T) \geq K$ while $K > n(T) \Rightarrow \int_{K-n(T)}^\infty (n(T) - K)\,\delta(u)du = 0$. Therefore, $C(T,T) = \max\{0, a\,(n(T) - K)\}$.

These results highlight the difference between pricing a call on revenue and a call on assets that can increase or decrease over time. If, for given x, $n + \mu - \bar{y}\Phi > K$, the option is sure to finish in-the-money since the left hand side is deterministic and moneyness can only increase with the occurrence of non-negative jumps. To break this down a bit further, note that $n(s)$ is already realized and $\mu - \bar{y}\Phi$ is deterministic drift.

In the case we say that the option is implicitly in-the-money and the pricing equation can be written

$$C(s,T) = ae^{-r(T-s)} \sum_{x=0}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \left(\int_0^{\infty} (n + \mu + (u - \bar{y}\Phi) - K)\, f_{u|x}(u) du \right) ` \quad (27)$$

$$= ae^{-r(T-s)} \left((n + \mu - \bar{y}\Phi - K) + \bar{y} \sum_{x=0}^{\infty} x e^{-\Phi} \frac{\Phi^x}{x!} \right) = ae^{-r(T-s)} (n + \mu - K) \,.$$

Volatility does not matter! When a revenue option is implicitly in-the-money, the payoff is linear and the problem becomes one of computing expected values.

Detailed analysis of the option pricing result depends on the distribution of $u(x) = \sum_{i=0}^{x(s,T)} y_i$, where where x is the random number of jumps, $y_0 = 0$, and y_i, $i = 1, 2,...$ is non-negative. The statistical problem is complicated by the fact that lognormal variates do not reproduce under addition and normal variates can be negative with positive probability. Thus, these distribution have practical and theoretical drawbacks in revenue jump processes.

An easy special case is that of constant jumps, $y_i = y$. Let $u = yx$ and $f_{u|x}(u) = \delta(u - yx)$ in equation (26) to get

$$C(s,T) = ae^{-r(T-s)} \sum_{x=0}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \left(\int_d^{\infty} (n + \mu + y(x - \Phi) - K)\, \delta(u - yx) du \right)$$

$$= ae^{-r(T-s)} \sum_{x=0}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \left((n + \mu + y(x - \Phi) - K) \right), \; yx \geq d \quad (28)$$

The integral equals $n + \mu + y(x - \Phi) - K$ when $yx \geq d$ because the point corresponding to non-zero mass (yx) is within integration limits. The condition $yx \geq d$ corresponds to the number of jump adopters required to place the call option in-the-money.

These results can also be cast in terms of the spot or futures value of an asset. Specifically, under the CAPM assumption, $V(s) = ae^{-r(T-s)}E\left(n(T)\right) = ae^{-r(T-s)}\left(n(s) + \mu(s,t)\right)$ and the value of a futures contract written on cumulative revenue at T is $F(s,T) = V(s)e^{r(T-s)}$. Equation (26) can therefore be written

$$C(s,T) = e^{-r(T-s)} \sum_{x=0}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \left(\int_d^{\infty} \left(F(s,T) + a(u - \bar{y}\Phi) - aK\right) f_{u|x}(u) du \right). \tag{29}$$

5.1. Gamma Distributed Jumps

The gamma distribution possess desirable properties since it reproduces under addition and has non-negative values. The density function of y, jump magnitude, is thus taken to be

$$f_y(y) = \frac{\left(\frac{y}{\beta}\right)^{\gamma-1} e^{\left(-\frac{y}{\beta}\right)}}{\beta\Gamma(\gamma)},\ y \geq 0,\ \beta > 0 \tag{30}$$

where γ and β are parameters. The moment generating function of y is

$$M_y(\theta) = \frac{1}{(1-\beta\theta)^{\gamma}}, \tag{31}$$

giving $E(y) = \gamma\beta$, $Var(y) = \gamma\beta^2$ and coefficient of variation, $CV = \frac{1}{\sqrt{\gamma}}$. For y_i iid, the moment generating of $u = \sum_{i=1}^{x} y_i$, conditional on x, is

$$M_{u|x}(\theta) = \frac{1}{(1-\beta\theta)^{x\gamma}}. \tag{32}$$

For fixed x, the distribution of u is thus gamma with parameters β and $\gamma^* = x\gamma$ and the cumulative distribution of u conditional on x is

$$F_{u|x}(u) = \int_0^u \frac{\left(\frac{t}{\beta}\right)^{x\gamma-1} e^{\left(-\frac{t}{\beta}\right)}}{\beta\Gamma_{x\gamma}} dt = 1 - \frac{\Gamma_{x\gamma}(u)}{\Gamma_{x\gamma}}, \tag{33}$$

where $\Gamma_{x\gamma} = \int_0^\infty t^{x\gamma-1}e^{-t}dt$ is the gamma function and $\Gamma_{x\gamma}(u) = \int_u^\infty f_{u|x}(t)dt$ is the "lower" incomplete gamma function.[9]

The marginal distribution of u can be characterized by moment generating functions and can be derived using iterated expectations. That is,

$$\begin{aligned} M_u(\theta) &= E_x\left(E_y(e^{\theta y})^x\right) = \sum_{x=0}^{\infty} e^{-\Phi}\frac{\Phi^x}{x!}\frac{1}{(1-\beta\theta)^{x\gamma}} \\ &= e^{-\Phi}\sum_{x=0}^{\infty}\frac{\left(\frac{\Phi}{(1-\beta\theta)^\gamma}\right)^x}{x!} = e^{\Phi((1-\beta\theta)^{-1}-1)}. \end{aligned} \tag{34}$$

The mean and variance of u follows directly and is given by $E(u) = \Phi\beta\gamma$ and $Var(u) = \Phi\gamma(\gamma+1)\beta^2$. Other useful results for the conditional distribution of u are

$$\int_d^\infty u f_{u|x}(u)du = \beta\gamma x\left(\frac{\Gamma_{\gamma x+1}\left(\frac{d}{\beta}\right)}{\Gamma_{\gamma x+1}}\right) \tag{35}$$

and

$$\begin{aligned} \int_d^\infty f_{u|x}(u)dy &= \frac{\Gamma_{\gamma x}\left(\frac{d}{\beta}\right)}{\Gamma_{\gamma x}},\ x = 1, 2, ... \\ &= \int_d^\infty \delta(u)du,\ x = 0 \end{aligned} \tag{36}$$

[9]The "upper" incomplete gamma function is defined as $\Gamma^*_\gamma(x) \equiv \int_0^x f dt$.We use the lower incomplete gamma function $\Gamma_\gamma(x) \equiv \int_x^\infty f dt$. Adding these functions give the gamma function, i.e., $\Gamma^*_\gamma(x) + \Gamma_\gamma(x) = \Gamma_\gamma$.

The option pricing formula corresponding to equation (26) is therefore

$$C(s,T) = ae^{-r(T-s)} \left(e^{-\Phi} \int_d^\infty (n + \mu - \beta\gamma\Phi - K)\, \delta(u) du \right) \tag{37}$$
$$+ae^{-r(T-s)} \sum_{x=1}^\infty \left(e^{-\Phi} \frac{\Phi^x}{x!} (n+\mu-K) \left(\frac{\Gamma_{x\gamma}\left(\frac{d}{\beta}\right)}{\Gamma_{x\gamma}} \right) + \beta \left(\frac{\Gamma_{x\gamma+1}\left(\frac{d}{\beta}\right) - \gamma\Phi\Gamma_{x\gamma}\left(\frac{d}{\beta}\right)}{\Gamma_{x\gamma}} \right) \right)$$

5.1.1. *Poisson Intensity Function*

A plausible choice for the Poisson intensity function is $\pi(t) = \rho f(t)$, where ρ is a constant and $f(t)$ is the Bass intensity function. The reasoning is that jumps are more likely to occur in times of peak deterministic intensity. More specifically, mean jump intensity is proportional to mean adopter intensity. Under this choice, the mean number of jumps in (s,T) is $\Phi(s,T) = \rho\,(F(T) - F(s))$ and $\mu(s,T) = mF(T)$. In the special case that $s = 0$, the call price formula is

$$C(0,T) = ce^{-rT} e^{\rho F} \left((m - \beta\gamma\rho)\, F - K \right) \int_d^\infty \delta(u) du \tag{38}$$
$$+ae^{-r(T-s)} \sum_{x=1}^\infty e^{-\rho F} \frac{(\rho F)^x}{x!} \left((n + mF - K) \left(\frac{\Gamma_{x\gamma}\left(\frac{d}{\beta}\right)}{\Gamma_{x\gamma}} \right) + \beta \left(\frac{\Gamma_{x\gamma+1}\left(\frac{d}{\beta}\right) - \gamma\rho F\Gamma_{x\gamma}\left(\frac{d}{\beta}\right)}{\Gamma_{x\gamma}} \right) \right),$$

where $F \equiv F(T)$.

5.1.2. *Probability of finishing in the money*

Revenue options that are implicitly in-the-money at time s finish in-the-money. But suppose the option at s is implicitly out-of-the money. Then $d > 0$ and the probability of finishing in-the-money is given by

$$\Pr ob(n(T) > K) = \sum_{x=1}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \int_d^{\infty} f_{u|x}(u) du = \sum_{x=1}^{\infty} e^{-\Phi} \frac{\Phi^x}{x!} \frac{\Gamma_{\gamma x}\left(\frac{d}{\beta}\right)}{\Gamma_{\gamma x}}. \tag{39}$$

6. SIMULATIONS

Simulations were performed using parameter estimates $m = 1.8048(10^7)$, $p = 0.337$ and $q = 1.1323$ from the movie, "Fellowship of the Ring" adoption process (next section). Parameters of the Poisson arrival process and jump magnitude were varied to provide a sensitivity analysis. As more data becomes available and the Kalman updating scheme is implemented, the Poisson parameters will also be estimated.

Table 1 gives results that vary by mean number of arrivals, $\Phi = \rho F(T)$, expected jump size, $E(y) = \gamma\beta$, and the coefficient of variation of jump size, $CV = \frac{1}{\sqrt{\gamma}}$. The first column with $CV = 0$ corresponds to fixed jump size. In fact, the fixed jump model ($CV = 0$) and $CV = 10^{-6}$ in the gamma model gives call values identical to the dollar.

Call option value increases monotonically with the coefficient of variation, the mean jump size and expected number of jumps. Because the option chosen is implicitly at-the-money, option value is linear homogeneous in expected jump size, e.g., note that options in Panel B are 10 times more expensive than options in Panel A. The total revenue expected in the first year for the Fellowship of the Ring, using estimates from our model is $12mF = \$93,679,550$. Options are implicitly at–the-money in the sense that the strike is set at the expected total revenue up through year one. The most expensive call option in the table, at $1,042,730 is 1.11% of expected year one revenue ($\frac{1{,}042{,}730}{93{,}679{,}550}$).

7. ESTIMATION

The main challenge the model poses to estimation is that the estimators are needed most when there is little or no revenue data. Movies earn the bulk of their revenues during the first few weekends after their release, as described, for example, in De Vany and Walls (1996). Therefore, we can expect the revenue options to be traded before the release of the movie and maybe for a short period after the release but not much longer.

7.1. Estimation With a Few Revenue Data Points

7.1.1. First-Moment Parameters: Forecast Loss Function

We write the expected value of equation (20) as,

$$\begin{aligned}\mathbb{E}(n(t)|t-1) - n(t-1) &= \mathbb{E}n(t-1,t) \\ &= \mathbb{E}(m|t-1)\int_{t-1}^{t} f(\tau)\,d\tau \\ &= \mathbb{E}(m|t-1)(F(t)-F(t-1)),\end{aligned}$$

where

$$F(t) = \frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}.$$

Using plug-in estimators $\hat{\theta} := (\hat{m},\, \hat{p},\, \hat{q})$ for $\theta := (m,\, p,\, q)$, we obtain:

$$\mathbb{E}n(t-1,t) = \hat{m}\left(\frac{1-e^{-(\hat{p}+\hat{q})t}}{1+\frac{\hat{q}}{\hat{p}}e^{-(\hat{p}+\hat{q})t}} - \frac{1-e^{-(\hat{p}+\hat{q})(t-1)}}{1+\frac{\hat{q}}{\hat{p}}e^{-(\hat{p}+\hat{q})(t-1)}}\right). \tag{40}$$

Observations on the adoption rate $n(t-1,t)$ are obtained from the observed cumulative revenue data $R(t)$ by dividing by an estimate $\hat{c}$ of the average ticket price c and then taking first differences.

The objective function to be minimized is a bilinear form in the distances $d_t := \hat{n}(t-1,t|\hat{\theta}) - n(t-1,t)$ of adoption forecasts implied by the parameter estimates from the observed adoption data. The optimization problem is, then,

$$\begin{aligned} \min_{\theta} \quad & d^T \Omega d \\ \text{s.t.} \quad & p, q > 0,\ m >> 0, \end{aligned} \tag{41}$$

where $d = (d_1, d_2, \ldots, d_T)^T$ and Ω is a weight matrix. We will use the identity on $\mathbb{R}^T$ for simplicity, that is, the objective is the sum of squares of the distances d_i. T is usually small. The first estimate of the three first-moment parameters m, p, and q of the process is available after $T = 3$ points of revenue data.

Sawhney and Eliashberg (1996) report that estimators of the class (41) have good qualities despite their small sample sizes because of the fact that the adoption dynamics are concentrated in the first few observations. They consider a parametrization

$$\mathbb{E}n(t-1,t) = \frac{m\lambda\gamma}{\lambda - \gamma}\left(e^{-\gamma t} - e^{-\lambda t}\right), \tag{42}$$

where λ is the intensity for the time an individual needs to decide whether to see a movie and γ is the intensity for the time this individual needs to act on this decision. The parameters λ and γ are assumed to be identical for all m individuals in the adopter population.

Figure 1 shows the estimation of model (40) on the first three observations of the adoption rate processes for the Lord of the Rings trilogy. In the figure, we use the legend S/E for Sawhney and Eliashberg (1996) and CHH for model (40). We also estimate the Sawhney and Eliashberg (1996) parametrization as a benchmark. The vertical axis is the total attendance. Parameter estimates for the Fellowship of the Rings attendance process (n) are $m = 1.8048(10^7)$, $p = 0.337$ and $q = 1.1323$. The figures show that the attendance rate peaks a zero and declines in an approximate exponential pattern after that.

We will use a database of movies released prior to the Lord of the Rings trilogy and estimate the revenue process parameters θ using maximum likelihood (A.1). Recording the characteristics (e.g., genre, country of release, budget, presence or absence of stars, number of screens...) of the movies in the database in the matrix X, we will regress the parameters on the movie characteristics in a panel model (A.2), similar to a Poisson regression model. Plugging in the characteristics of the first part of the trilogy yields a point estimate of the parameters of the revenue process of the movie before any revenue data are available. After the release, as the first revenue observations become available, we will update the estimate of the parameters θ using an extended Kalman-filter approach. We can iterate this procedure for the second and the third part of the trilogy, including information from the earlier parts into the historical database. Using the initial and the updated estimates of the parameter vector, we can calculate the value of revenue options on the three movies at different stages in the release process and for different revenue thresholds. Finally, we will compare our estimates with the historical box office data for the trilogy.

Remaining empirical work includes estimating second moment parameters and parameter updating using a Kalman filtering scheme. An outline of the approach is given in the appendix.

8. SUMMARY AND CONCLUSIONS

In this paper we have examined the pricing of options on a non-decreasing underlying. These instruments have surfaced in the form of contingent claims on the revenue streams of movies, concert tours, and music recordings. The options are typically sold before any revenue is generated; hence, at that time there is no observable value of the underlying. Moreover, the non-decreasing revenue stream poses significant challenges to modeling the stochastic evolution of the underlying. We develop a stochastic model based on the notion that an individual's decision to purchase the product is driven by two factors: the systematic influence of others who have already purchased the product and an idiosyncratic effect independent of the actions of others. The stochastic component of the model is captured with a Poisson jump process, which is uncorrelated with the market factor. We derive boundary conditions, put-call parity, results for early exercise of American options, and of course option pricing equations.

Further challenges are encountered in implementing the model. To obtain reasonable estimates of the five input parameters required in the model, we build an econometric model of movie revenues. These revenues are well-known to have a significant concentration of uncertainty in the near term, which is resolved very quickly. Hence, the parameters for pricing the option before the movie is released should be quite different from those for pricing the movie after the first few weeks of its life. In the first case, we use a panel regression model to estimate the parameters based on characteristics of the movie, the timing of release, and marketing. In the second case, the first few weeks of revenue data for the movie provide significant information and are used with a Kalman filter to update the parameter estimates. Preliminary empirical tests are conducted using weekly data on revenues generated from recent movies.

Hollywood has shown considerable ingenuity in partnering with Wall Street to offer these instruments, as well as securitized equity claims on movie revenues. The implications of our paper are important for the pricing of these types of options but could also be useful for other possible structures. For example, the owner of an oil field might sell a call option on oil revenues where the exercise price is the production cost. The value of such an option would be driven by two factors, a non-decreasing but stochastic stream of output and a stochastic price. Options on revenue streams would seem to be a natural component of real option theory, where claims are often a function of revenues rather than traded and easily valued assets.

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APPENDIX

Second-Moment-Parameters: Maximum Likelihood

In the sections that follow, we outline an approach for estimating second moment parameters. For option pricing, the volatility parameters α and ρ are of main importance. From (20), where $y = \alpha$ constant, we have that

$$
\begin{aligned}
\mathbb{P}(n(T) - n(s)) &= \mathbb{P}\left(x(s,T) = \left[\frac{k}{\alpha} + \int_s^T \pi(t)dt - m \int_s^T f(t)dt \right] \right) \\
&= \mathbb{P}\left(x(s,T) = \left[\frac{k}{\alpha} + (\rho - \frac{m}{\alpha})(F(T) - F(s)) \right] \right) \\
&= \exp\left(- \int_s^T \pi(t)dt \right) \frac{\left(\int_s^T \pi(t)dt \right)^{\kappa}}{\kappa!} \\
&= \exp\{-\rho(F(T) - F(s))\} \frac{\rho^{\kappa}(F(T) - F(s))^{\kappa}}{\kappa!},
\end{aligned}
$$

where

$$
\kappa = \left[\frac{k}{\alpha} + (\rho - \frac{m}{\alpha})(F(T) - F(s)) \right],
$$

and $[x] = \max\{n \in \mathbb{N} | n \leq x\}$. Then, given observations of the adoption process $n(t-1,t) =: k_t$, the log-likelihood function is given by

$$
\begin{aligned}
L(\theta) &= \sum_{t=2}^{T} \log \mathbb{P}\{n(t) - n(t-1) = n(t-1,t) = k_t\}, \qquad \text{(A.1)} \\
&= \sum_{t=2}^{T} \left(\log \frac{\rho^{\kappa(t)}(F(t) - F(t-1))^{\kappa(t)}}{\kappa(t)!} - \rho(F(t) - F(t-1)) \right),
\end{aligned}
$$

where

$$
\kappa(t) = \left[\frac{k_t}{\alpha} + (\rho - \frac{m}{\alpha})(F(t) - F(t-1)) \right].
$$

Estimation Without Revenue Data

Estimate of the First Revenue Data Point

If there are no revenue and thereby adoption data available, the adoption process must be forecast using historical data from other movies. A standard approach is the Poisson regression model, in which the log of a Poisson density parameter is modelled as a linear function of a set of explanatory variables. The variables discussed in the context of movies are, to name a few, genre, country of release, budget, presence or absence of stars, number of screens, and a time trend accounting for the sharp drop after the first few weekends (Jones and Ritz 1991, Sawhney and Eliashberg 1996, Neelamegham and Chintagunta 1999). Neelamegham and Chintagunta (1999) suggest a Bayesian hierarchical model to obtain a density forecast of the Poisson parameter. We use a panel regression model for the log of the five parameters of the adoption process considered here, similar to the Poisson regression. From a set of revenues series of k other movies, the parameter vectors θ_i, $i = 1, \ldots, k$ are estimated by maximizing (A.1). Then, the log of the k estimates of each parameter are regressed on the explanatory variables. The estimation equations are, thus,

$$\log \hat{\theta}_i = \log \begin{bmatrix} \hat{m}_i \\ \hat{p}_i \\ \hat{q}_i \\ \hat{\alpha}_i \\ \hat{\rho}_i \end{bmatrix} = \begin{bmatrix} \xi_{i,t}\beta^m + \eta^m_{i,t} \\ \xi_{i,t}\beta^p + \eta^p_{i,t} \\ \xi_{i,t}\beta^q + \eta^q_{i,t} \\ \xi_{i,t}\beta^\alpha + \eta^\alpha_{i,t} \\ \xi_{i,t}\beta^\rho + \eta^\rho_{i,t} \end{bmatrix}, \quad \begin{array}{c} i = 1, \ldots, k, \\ t = 1, \ldots, T_i. \end{array} \tag{A.2}$$

where $\eta^{(\cdot)}_{i,t}$ is white noise with mean zero and variance σ^2_η. We assume that there is no covariance across parameters and across movies and that there is no serial correlation. The $(1 \times K)$ regressor vector $\xi_{i,t}$ for i and t fix consists of K explanatory variables from the list stated above. The time index t runs from 1 to T_i, the life span of movie i. We do not write a time index to the estimates of θ because these do not vary with time. Among the explanatory variables some will not vary with time (genre, presence of stars, ...) and some will (number of screens, time trend, ...). Plugging in the ξ characteristics of the movie for which the initial adoption is to be estimated yields a point estimate for θ at time $t = 0$, that is, before revenue data are available.

Update Scheme for Incoming Revenue Observations

The estimates of the parameters θ of the adoption process of the new movie have to be updated as the first observations of the process come in. To this end, we pursue a Kalman filter approach. As adoption process, we employ (**??**) with $\delta = 0$ so that $y = \alpha$ is constant. Since the (log) parameters enter the adoption process non-linearly, the Kalman filter equations use the first order Taylor expansion of the adoption process (extended Kalman filter, Harvey 1989, pp 160ff).

The State Space Model

The non-linear state space model is specified as

$$y_t = z_t(\varsigma_t) + \varepsilon_t, \qquad \text{(A.3)}$$

$$\varsigma_t = T_t \varsigma_{t-1} + R\eta_t, \qquad \text{(A.4)}$$

where (A.3) is the *measurement equation* specifying the process of observations

$$z_t(\varsigma_t) = \log n(t-1,t) = \log\left[(m-\alpha\rho)(F(t)-F(t-1)) + \alpha x(t-1,t)\right]. \tag{A.5}$$

The error processes ε and η are assumed to have zero-mean and variance and covariance matrix, respectively,

$$\mathbb{E}\varepsilon_t^2 = h,$$
$$\mathbb{E}\eta_t\eta_t^T = Q.$$

Equation (A.4) is the *transition equation* governing the *state vector* process ς_t, containing the parameters $\log\theta$, all hyperparameters $\beta^{(\cdot)}$, and the number of jumps $x(t-1,t)$. Even though the parameters and hyperparameters are constant in the data-generating process, they can be described in a transition equation where they carry a time subscript t (Harvey 1989, p 104). Let vec $\hat{\beta}_t$ be the stacked column vector of hyperparameters in the transition equation:

$$\text{vec}\,\hat{\beta}_t = \left((\hat{\beta}_t^m)^T,\, (\hat{\beta}_t^p)^T,\, (\hat{\beta}_t^q)^T,\, (\hat{\beta}_t^\alpha)^T,\, (\hat{\beta}_t^\rho)^T\right)^T \in \mathbb{R}^{5K\times 1}$$

Denote $0(n_1, n_2)$ as the zero matrix with n_1 rows and n_2 columns, and $I(n)$ as the n-dimensional identity matrix. Then, the transition equation is

$$\varsigma_t = \begin{bmatrix} \log\theta_t \\ \text{vec } \beta_t \\ x(t-1,t) \end{bmatrix} = T_t \begin{bmatrix} \log\theta_{t-1} \\ \text{vec } \beta_{t-1} \\ x(t-2,t-1) \end{bmatrix} + I(5{+}5K{+}1) \begin{bmatrix} \eta_t^m \\ \eta_t^p \\ \eta_t^q \\ \eta_t^\alpha \\ \eta_t^\rho \\ 0(5K+1,1) \end{bmatrix}, \tag{A.6}$$

where

$$T_t = \begin{bmatrix} 0(1,5) & & \xi_t & 0(1,4K) & 0 \\ 0(1,5) & 0(1,K) & \xi_t & 0(1,3K) & 0 \\ 0(1,5) & 0(1,2K) & \xi_t & 0(1,2K) & 0 \\ 0(1,5) & 0(1,3K) & \xi_t & 0(1,1K) & 0 \\ 0(1,5) & 0(1,4K) & \xi_t & & 0 \\ 0(5K,5) & & I(5K) & & 0(5K,1) \\ & & 0(1,5+5K+1) & & \end{bmatrix},$$

$T_t \in \mathbb{R}^{(5+5K+1)\times(5+5K+1)}$. Thus, R in equation (A.4) is given by $I(5 + 5K + 1)$ and η_t in equation (A.4) is given by the error vector in equation (A.6).

In the transition equation, the parameters $\log\theta$ are modelled as a linear combination of ς with hyperparameters β as coefficients plus noise η, the hyperparameters are simply stated as identity, and the number x of jumps is not related to its lagged value since the jumps of a Poisson process in different time intervals are independent.

Linearization

Next, we need to linearize $z_t(\varsigma_t)$ by means of a first-order Taylor expansion. The gradient of $z_t(\varsigma_t)$ is given by

$$\text{grad } z_t(\varsigma_t) = \left[\frac{\partial z_t}{\partial \log \theta_t^T}, \frac{\partial z_t}{\partial \text{ vec } \beta_t^T}, \frac{\partial z_t}{\partial x(t-1,t)}\right] \tag{A.7}$$

From (A.5), suppressing the time subscript of the parameters, and writing $(\cdot)$ for (m), (p), (q), (α) or (ρ), we have that

$$\begin{aligned}
\frac{\partial z_t}{\partial \log m} &= \frac{m(F(t)-F(t-1))}{n(t-1,t)}, \\
\frac{\partial z_t}{\partial \log p} &= \frac{(m-\rho\alpha)(F(t)-F(t-1))}{n(t-1,t)} \frac{\partial \log(F(t)-F(t-1))}{\partial \log p}, \\
\frac{\partial z_t}{\partial \log q} &= \frac{(m-\rho\alpha)(F(t)-F(t-1))}{n(t-1,t)} \frac{\partial \log(F(t)-F(t-1))}{\partial \log q}, \\
\frac{\partial z_t}{\partial \log \alpha} &= \frac{-\rho\alpha(F(t)-F(t-1))+\alpha x(t-1,t)}{n(t-1,t)}, \\
\frac{\partial z_t}{\partial \log \rho} &= \frac{-\rho\alpha(F(t)-F(t-1))}{n(t-1,t)}, \\
\frac{\partial z_t}{\partial \left(\beta^{(\cdot)}\right)^T} &= \frac{\partial z_t}{\partial \log(\cdot)} \frac{\partial \log(\cdot)}{\partial \left(\beta^{(\cdot)}\right)^T} = \frac{\partial z_t}{\partial \log(\cdot)} \xi_t, \\
\frac{\partial z_t}{\partial x(t-1,t)} &= \frac{\alpha}{n(t-1,t)},
\end{aligned} \tag{A.8}$$

where

$$\frac{\partial \log(F(t)-F(t-1))}{\partial \log p} =$$
$$\frac{pe^{p+q}}{e^{p+q}-1} - \frac{q}{p+q} - pt - \frac{qte^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}} - \frac{q(t-1)e^{-(p+q)(t-1)}}{1+\frac{q}{p}e^{-(p+q)(t-1)}},$$
$$\frac{\partial \log(F(t)-F(t-1))}{\partial \log q} =$$
$$\frac{q}{p+q} - qt + \frac{qe^{p+q}}{e^{p+q}-1} + \frac{\frac{q^2}{p}te^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}} + \frac{\frac{q^2}{p}(t-1)e^{-(p+q)(t-1)}}{1+\frac{q}{p}e^{-(p+q)(t-1)}}.$$

Denote the estimate of the state vector ς_t as s_t and the estimate of the gradient (A.7) of z_t by $\hat{Z}_t$:

$$\hat{Z}_t = \frac{\partial z_t(\varsigma_t)}{\partial \varsigma_t^T}.$$

Expand the first order Taylor series approximation in the point $s_{t|t-1}$ of the forecast of s_t given information through $t-1$. Then, the linearized state space model is given by

$$y_t \doteq z_t(s_{t|t-1}) + \hat{Z}_t(\varsigma_t - s_{t|t-1}) + \varepsilon_t, \tag{A.9}$$
$$\varsigma_t = T_t\varsigma_{t-1} + R\eta_t, \tag{A.10}$$

The Kalman filter equations are now determined (Harvey 1989, p 161). They consist of the *prediction equations* for the state vector and the covariance matrix P of the state vector

$$s_{t|t-1} = T_t s_{t-1},$$
$$P_{t|t-1} = T_t P_{t-1} T_t^T + RQR,$$

and the *update equations* for both

$$s_t = s_{t|t-1} + P_{t|t-1}\hat{Z}_t^T\hat{F}_t^{-1}(y_t - z_t(s_{t|t-1})),$$

$$P_t = P_{t|t-1} - P_{t|t-1}\hat{Z}_t^T\hat{F}_t^{-1}\hat{Z}_t P_{t|t-1},$$

$$\hat{F}_t = \hat{Z}_t P_{t|t-1}\hat{Z}_t^T + h.$$

The updated estimation s_t of the state vector ς_t then allows to update the estimate $\hat{n}(t-1,t)$ of the adoption process.

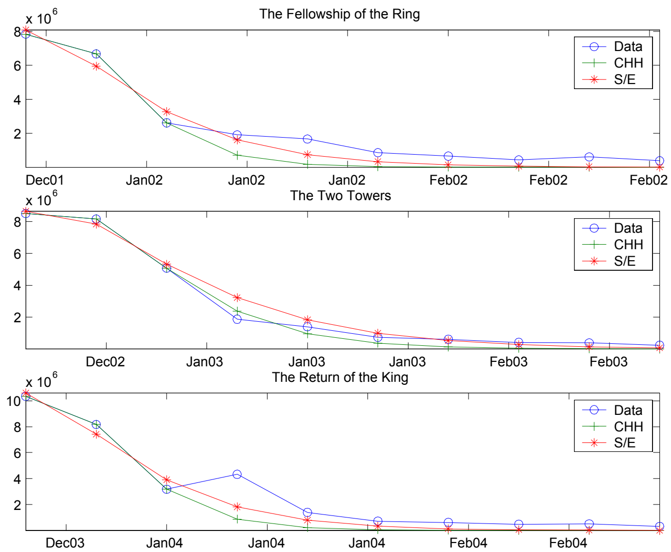

Figure 1: Adoption Rate Process for the Lord of the Rings Triology

S/E is the Sawhney and Eliashberg model and CHH is the model of equation (40).

TABLE 1.

Call Prices and Jump Process Parameters

ρF	$CV = \frac{1}{\sqrt{\gamma}}$				
	0	0.20	0.40	0.60	0.80
	Panel A: Mean Jump Magnitude:$\gamma\beta = 1,000$				
20	$20,281	20,767	21,926	23,727	26,032
40	28,741	29,370	31,017	33,571	36,849
60	35,225	35,971	37,986	41,123	45,145
80	40,688	41,536	43,864	47,488	52,137
	Panel B: Mean Jump Magnitude: $\gamma\beta = 10,000$				
20	$202,807	207,669	21,9259	237,273	260,319
40	287,410	293,700	31,0137	335,714	368,490
60	352,248	359,713	37,9862	411,230	451,446
80	406,883	415,364	43,8641	474,885	521,365
	Panel C: Mean Jump Magnitude: $\gamma\beta = 20,000$				
20	$405,613	415,338	438,517	474,546	520,637
40	574,820	587,401	620,274	671,429	736,979
60	704,497	719,427	759,725	822,459	902,891
80	813,765	830,728	877,282	949,769	1,042,730

The mean number of jumps is ρF. The mean jump magnitude and coefficient of variation of y is $\gamma\beta$ and $\frac{1}{\sqrt{\gamma}}$. $CV = 0$ corresponds to fixed jump size. The option expires in one year and is implicitly at-the-money since the strike is set equal to expected revenue. The option expiresin one year and the risk-free rate is 5%.