RETURN GUARANTEES WITH DELAYED PAYMENT

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Abstract. A unit–linked insurance contract can be formulated in terms of a guaranteed amount together with a fraction of a positive excess return of a benchmark portfolio. Normally, the excess return is determined annually and accumulated until the maturity of the contract. The accumulation factor which is granted with respect to the delayed payments can either be deterministic or equal to the (stochastic) bank account. It turns out that the common choice of a deterministic accumulation factor gives rise to problems concerning the pricing and the risk management of the insurance contract.

Keywords: *Periodic return guarantee, delayed payment option, fair contract, defined–contribution pension plans, life–insurance, uncertain volatility, conservative pricing, robust hedging, model misspecification, model risk*

JEL–codes: *G13, G22, G23*
Insurance Branch Codes: *IB13, IB81*

1. Introduction

Past difficulties with the financial derivatives embedded in life insurance contracts are in part due to the fact that the classical actuarial approach, i.e. valuing liabilities by discounting expected payouts with a fixed interest rate, is not applicable. Traditionally one could justify the use of expected payouts by the law of large numbers, i.e. for a company with a large number of policies outstanding, the mortality tables reflect nearly deterministic proportions rather than probabilities. This is not true for claims depending on one or few underlying financial variables. Rather, an approach based on arbitrage arguments of the type pioneered by Black and Scholes (1973a) is required. Thus "fair valuation" becomes "pricing by arbitrage" where the embedded options are concerned. This was recognised early on by Brennan and Schwartz (1976, 1979) and Boyle and Schwartz (1977).

We consider a contract (or savings plan) where the payoff can be formulated in terms of a guaranteed amount together with a fraction of a positive excess return on the basis of a benchmark portfolio. In particular, the excess return is linked directly to the level of a product which is traded at a financial market, such as a mutual fund, a certain stock, a foreign currency, etc. Important examples involving such a contract situation are known as *unit–linked* insurance contracts with minimum return guarantees. References to unit–linked contracts with minimum return guarantees are Bacinello and Ortu (1993), Nielsen and Sandmann (1996, 2002) , Boyle and Hardy (1997), Grosen and Jørgensen (1997), Bacinello (2001) and Miltersen and Persson (2003). These papers are concerned with the valuation of different types of minimum return guarantees. They all hold in common the use of martingale pricing theory based on the works of Harrison and Kreps (1979) and Harrison and Pliska (1981).

However, the above literature is mainly concerned with the correct valuation. Along the lines of Mahayni and Schlögl (2003) we are also concerned with the risk management under model misspecification and model risk. The hedging needs close attention for various reasons. The risk management affects the default of the underwriter as well as the benefits to the insured. The contracts have a long time to maturity which also gives a hedging problem. Thus, the robustness of hedging strategies against model misspecification can be seen as a necessary requirement.

The main focus of this paper is the delayed payment structure of the return guarantees. For example, in Germany the excess return of a unit–linked insurance contract is delayed until contract maturity. In particular, it is often deferred without interest. In general, an interest rate can be granted for the lag–time. This can be modelled in terms of an accumulation factor which can either be deterministic or stochastic in the sense that it coincides with the bank account prescribed by the future spot rates.

In particular, the payoff of the contract under consideration depends on the decision of how the time delay is honored in terms of an accumulation factor. In the case of a deterministic factor the interest rate granted for the time delay is known at the inception of the contract. One motivation of this might be to avoid any additional randomness introduced by a stochastic accumulation factor. An insurance company might be tempted to think that it is easier to handle a contract where the accumulation factor is deterministic, in particular if the size is fixed in a conservative manner. In the following, we use standard theory from financial economics to show that just the opposite is true.

Although the main motivation of this paper stems from the analysis of unit–linked life insurance contracts, the results are also valid for investment strategies with minimum return guarantees. In a more general or abstract setup our results become of principal importance as soon as the final payoff can be decomposed into a sum of periodic payments which are delayed until the maturity of the contract. This is a common feature of many investment strategies actually offered to investors.

In financial terms, the embedded derivatives are given by forward starting near the money options where the payoff is delayed to the maturity of the contract. It is shown that a deterministic accumulation factor implies an additional convexity correction. This is explained by the theory of *pricing by no arbitrage*. Within a complete financial market, the no–arbitrage price of a contingent claim can be expressed as the expectation of the discounted payoff under the so–called risk neutral measure. In the case of a stochastic bank account, the price of the time delay is exactly compensated. Therefore, the embedded option is easily interpreted as an option without time delay. In the case of a deterministic accumulation factor, the size of the additional convexity adjustment is proportional to the initial forward rate. Furthermore, the excess return is a non–linear function of the benchmark index. Therefore, the convexity adjustment also depends on the correlation of the excess returns and the future spot rates. The higher the correlation, the higher the required compensation will be.

In order to analyze the hedging and robust hedging of the contracts, it is convenient to interpret the embedded options as exchange options. The underlying *pseudo* assets depend on the choice of the accumulation factor. If the accumulation factor is deterministic, these assets are neither necessarily traded nor necessarily observable. With respect to a complete market, for example in a Black/Scholes–type setup, these assets can easily be synthesized. In the case of a stochastic bank account, the resulting strategy for the option is still a Black/Scholes–type strategy. This is not the case if the accumulation factor is deterministic. The well known robustness results of Black/Scholes–type strategies are not valid for the hedging of a delayed payment option with a deterministic accumulation factor. This is even more important with regard to the high degree of model risk due to the long time to maturity of the financial products under consideration.

The paper is organized as follows. Sec.2 gives the contract design. Sec.3 uses the rich toolbox of financial economics for pricing the insurance contract under consideration, i.e. the delayed payment options. Sec.4 analyzes the implications for the fair contract parameters. These are the participation rate specifying the fraction of excess return granted to the insured and the guaranteed rate determining the guaranteed amount. It turns out that a deterministic accumulation factor is not necessarily compatible with a fair participation rate between zero and one which is normally the intention of the contract design. Of course, this is due to the *loss* incurred by the delayed payments. Sec.5 studies the deterministic accumulation factor which offsets the effect of the time delay. In particular, the factor is represented as an adjustment of the forward price of a zero coupon bond where the direction of the adjustment is determined by the correlation of the spot rate and asset prices. The risk management of the insurance contracts under consideration is analyzed in sec.6. In particular, the implications of so–called model risk are discussed. Finally, sec.7 concludes the paper.

2. Contract specification

For simplicity, we assume that the investor pays a constant periodic premium A. $\tau = \{t_0, \dots, t_{N-1}, t_N\}$ denotes the set of reference dates. The maturity date of the contract is indicated by t_N. With respect to an insurance contract, the time between two trading dates is equal to one month or one year. In this case, one can think of t_N as the "time of retirement" when premium payments cease and in the simplest case, the accumulated funds are paid out as a lump sum. In particular, the insured pays the amount A at each time t_i $(i = 0, \dots, N-1)$.

The payoff which the policy holder receives at time t_N is given in terms of a guaranteed part G together with a bonus account B. The guaranteed part G resembles an insurance account where each premium A is invested according to a guaranteed rate g, i.e.

$$(1) \qquad G_{t_N} = e^{g(t_N - t_{N-1})} \tilde{A}_{t_{N-1}}, \text{ where } \tilde{A}_{t_i} := \sum_{j=0}^{i} A e^{g(t_i - t_j)},\ i = 0, 1, \dots, N-1.$$

The bonus account B is determined by the excess returns which are based on a benchmark portfolio. In particular, we assume that the surplus which is observed from t_i up to t_{i+1} for $i = 0, \dots, N-1$ is premised on the comparison of two investment alternatives. The premium which is paid until t_i, i.e. the insurance account $\tilde{A}_{t_i}$, can either be used to buy the benchmark index S or to invest according to the guaranteed rate g. Thus, the excess return is composed of the positive part of the difference of

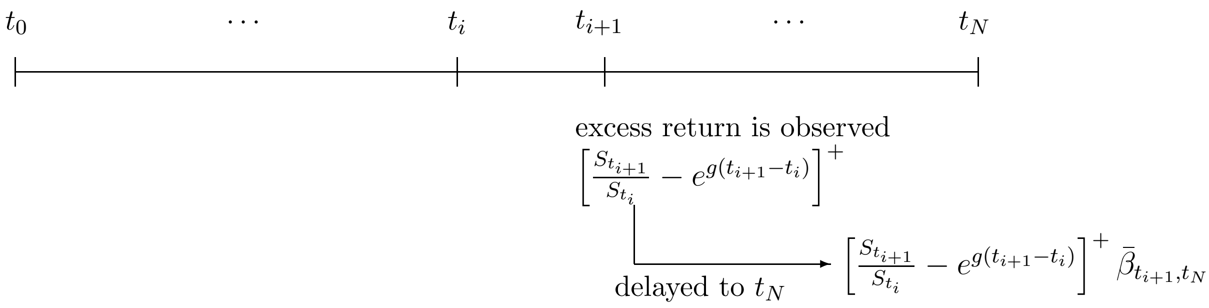

FIGURE 1. Delayed payment.

$\frac{\tilde{A}_{t_i}}{S_{t_i}} S_{t_{i+1}}$ and $\tilde{A}_{t_i} e^{g(t_{i+1}-t_i)}$, i.e. the t_i up to t_{i+1} excess return is given by

$$\tilde{A}_{t_i} \left[\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right]^+$$

It is worth mentioning, that in Germany the excess return is, although observed at t_{i+1}, delayed until the contract maturity t_N. In particular, it is often deferred without deigning an interest rate. However, we assume that an interest rate can be granted for the lag–time. This is modelled in terms of an accumulation factor $\bar{\beta}_{t_{i+1},t_N}$, c.f. Figure 1. Before we comment on $\bar{\beta}$, we finish the contract specification by the observation that only a fraction α $(0 \leq \alpha \leq 1)$, the so called *participation rate*, of the excess returns is granted to the customer. Thus the payoff at time t_N in terms of the bonus account amounts to

$$B_{t_N} := \alpha \sum_{i=0}^{N-1} \tilde{A}_{t_i} \bar{\beta}_{t_{i+1},t_N} \left[\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right]^+ . \tag{2}$$

To sum up, the investment contract is defined by the contributions of the policy holder and the insurer, i.e. the periodic premium A and the final payoff $I(t_N)$ which is defined by

$$I(t_N) := G_{t_N} + B_{t_N} \tag{3}$$

where G_{t_N} and B_{t_N} are given by equation (1) and equation (2).

Notice that the reference portfolio S is fixed a priori. Thus the payoff granted at maturity t_N depends on the one hand on the choice of the accumulation factor $\bar{\beta}$ and on the other hand on the contract parameters α and g.

A few comments are necessary in order to explain the significance of the accumulation factor $\bar{\beta}$. Basically, it can either be determined at the initialization of the contract, or it can be based on the future spot rates. Thus, with respect to the accumulation factor $\bar{\beta}$, we like to consider the following basic scenarios:

(a) $\bar{\beta}$ coincides with the stochastic bank account, i.e.

$$\bar{\beta}_{t_i,t} = \exp\left\{\int_{t_i}^{t} r_u \, du\right\} \quad \forall \; t_i < t, \; \forall \, t \leq T,$$

where $r = (r_t)_{0 \leq t \leq t_N}$ denotes the continuously compounded spot rate.

(b) $\bar{\beta}_{t_i,t}$ can be expressed by a guaranteed interest which is not necessarily equal to g but given by a deterministic function In particular, this includes no interest, i.e. $\bar{\beta}_{t_i,t} = 1$.

As our paper will show, the choice of $\bar{\beta}$ plays an important role in the valuation and risk management of the insurance contract. In order to distinguish between these scenarios, we will use the convention that in scenario (a) an accumulation factor is given in terms of the stochastic bank account, while scenario (b) denotes the deterministic accumulation factor including the case of no interest.

The first question which presents itself is how to specify a *fair* contract, i.e. how to specify the *fair* contract parameters α^* and g^* for a given periodic premium A. The so–called equivalence principle states that a contract is fair if the present value of the contributions is equal to the present value of the benefits. Let $D(t_0, t)$ denote the discount factor with time to maturity $t - t_0$. The present value of the contributions of the policy holder is given by

$$A \sum_{i=0}^{N-1} D(t_0, t_i).$$

The benefits to the insured, i.e. $I(t_N)$ given by equation (3), consist of a deterministic part with present value

$$D(t_0, t_N)\tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})}$$

and a random payoff B which is given in terms of call–options on the benchmark index S.[1] Therefore, to calculate the fair price of the embedded options, we use standard theory from financial economics which is based on arbitrage arguments. Let $C_t^{(i)}$ $(i = 0, \ldots, N-1)$ denote the *arbitrage–free* price a $t \in [0, t_N]$ of an European contingent claim where the payoff at contract maturity $T = t_N$ is given by

$$C_{t_N}^{(i)} := \bar{\beta}_{t_{i+1},t_N} \left[\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i-1} - t_i)} \right]^+ . \tag{4}$$

It follows that a fair contract specification (α, g) depends on the arbitrage free t_0–option prices $C_{t_0}^{(i)}$ and is given by the solution of

$$A \sum_{i=0}^{N-1} D(t_0, t_i) = D(t_0, t_N)\tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})} + \alpha \sum_{i=0}^{N-1} \tilde{A}_{t_i} C_{t_0}^{(i)}.$$

It is worth emphasizing that the determination of "fair" prices according to standard theory is justified by the existence of hedging strategies, i.e. self–financing trading strategies which replicate the payoff under consideration at maturity. This implies that the fair price is given by the amount necessary to initialize the self–financing replicating strategy. If such a perfect hedge is used, the guarantees of the insurance contract can be honored with probability one. Note that these strategies are dynamic and rely on the embedded option prices at each $t \in [0, t_N]$. In order to gain insight into the hedging possibilities, it is not enough to specify the t_0–price of the insurance

[1]Notice that the option features which are embedded into the insurance contract differ from standard option features. In financial terms, the embedded derivatives are given by forward starting options where the payoff is delayed to the maturity of the insurance contract.

contract needed to design a fair contract, but to consider the price process $(I_t)_{0\le t\le t_N}$ where

$$I(t) = D(t,t_N)\left(\tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})}\right) + \alpha \sum_{i=0}^{N-1} \tilde{A}_{t_i} C_t^{(i)}.$$

To start the analysis of the fair contract specification we have to determine the price process $(C_t^{(i)})_{0\le t\le t_N}$ with respect to a complete financial market. We have to distinguish between a stochastic and a deterministic accumulation factor.

3. Complete financial market model

In the following, we assume a complete and arbitrage–free financial market model under interest rate risk where the dynamic of the index price process S as well as the dynamics of the zero coupon bonds $D(.,\bar{t})$ paying one monetary unit at maturity $\bar{t} \in [0,T]$ are lognormal.

Definition 3.1 (Lognormal Process). We call a stochastic process $(Z_t)_{0\le t\le T}$ *lognormal* iff it is a solution of

$$dZ_t = Z_t\left(\mu_t dt + \sigma_Z(t) dW_t\right)$$

with *deterministic* dispersion coefficient $\sigma_Z : [0,T[\to \mathbb{R}_+^d$.

Thus, the index dynamic is modeled along the lines of Black and Scholes (1973b), the interest rate dynamic is given by a Gauss–Markov Heath, Jarrow and Morton (1992) model. In particular, we assume the existence of a uniquely defined martigale measure P^* such that

$$dS_t = S_t\left(r_t dt + \sigma_S(t) dW_t^*\right) \tag{5}$$

$$dD(t,\bar{t}) = D(t,\bar{t})\left(r_t dt + \sigma_{\bar{t}}(t) dW_t^*\right) \tag{6}$$

where W^* denotes a d–dimensional Brownian Motion with respect to P^*, and σ_S and $\sigma_{\bar{t}}$ satisfy the usual regularity conditions. The volatility of the forward price process of the bond with maturity T_2 with respect to the T_1–bond is given by $\sigma_{T_2}(t) - \sigma_{T_1}(t)$.[2] Analoguous, the forward volatility of the index S with respect to the $\bar{t}$–bond is given by $\sigma_S(t) - \sigma_{\bar{t}}(t)$. In the following, we will simply refer to the volatility of the quotient process $\frac{X}{Y}$ as the forward volatility and use the shorter notation $\sigma_{X,Y}$ for $\sigma_X - \sigma_Y$.

Recall that $C_t^{(i)}$ $(i = 0,\ldots,N-1)$ denotes the *arbitrage–free* price at $t \in [0,t_N]$ of the embedded option due to the excess return from t_i to t_{i+1} where the payment is lagged to t_N, i.e. an European contingent claim where the payoff at the contract maturity $T = t_N$ is given by equation (4). In the following, we consider a fixed reference period $[t_i, t_{i+1}]$ $(i \in \{0,\ldots,N-1\})$ and simplify our notation by using C instead of $C^{(i)}$ as a symbol for the relevant embedded option. We distinguish between a stochastic accumulation factor, i.e. scenario (a) and a deterministic accumulation factor, i.e. scenario (b), and denote the option prices with $C^{(a)}$ and $C^{(b)}$, respectively. For $t \in]t_{i+1}, t_N]$ we can immediately write down the price process in both cases since

[2] Notice that the quotient process of two lognormal processes is lognormal as well. In particular, the local volatility of the quotient process is simply given by the difference of the local volatilities.

the excess return is already known, i.e. we have

$$
\begin{aligned}
C_t^{(a)} &= \exp\left\{\int_{t_{i+1}}^{t} r_u\,du\right\}\left[\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right]^+ \quad \text{for all } t \in]t_{i+1}, t_N], \\
C_t^{(b)} &= D(t,t_N)\bar\beta_{t_{i+1},t_N}\left[\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right]^+ \quad \text{for all } t \in]t_{i+1}, t_N].
\end{aligned}
$$

where β_{t_{i+1},t_N} is deterministic in scenario (b). For $t \in [t_0, t_{i+1}]$, it is intuitively clear that the option which is included in the insurance contract can be interpreted as an exchange option. More precisely, we can use the interpretation of the following proposition.

Proposition 3.2. *For $t \in [t_0, t_{i+1}]$ and $w \in \{a, b\}$, the t–price of the embedded option C is given by the t–price of an option to exchange two (pseudo–) assets $X^{(w)} = X$ and $Y^{(w)} = Y$ where the payoff at t_{i+1} is given by $[X_{t_{i+1}}^{(w)} - Y_{t_{i+1}}^{(w)}]^+$. In particular, the processes $X = (X_t)_{0\le t\le t_{i+1}}$ and $Y = (Y_t)_{0\le t\le t_{i+1}}$ are in the case of stochastic bank account given by*

$$
\begin{aligned}
X_t^{(a)} &= D(t,t_i)\,1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}}\,1_{\{t\in[t_i,t_{i+1}]\}} \\
Y_t^{(a)} &= D(t,t_{i+1})e^{g(t_{i+1}-t_i)}\,1_{\{t\in[t_0,t_{i+1}]\}}
\end{aligned}
$$

and in the case of a deterministic accumulation factor $\bar\beta$ they are determined by

$$
\begin{aligned}
X_t^{(b)} &= \bar\beta_{t_{i+1},t_N}\frac{D(t,t_N)}{D(t,t_{i+1})}\,c(t,t_i,t_{i+1})\,X_t^{(a)} \\
Y_t^{(b)} &= \bar\beta_{t_{i+1},t_N}\frac{D(t,t_N)}{D(t,t_{i+1})}\,Y_t^{(a)}
\end{aligned}
$$

where the proportionality factor $c(t,t_i,t_{i+1})$ is defined by

$$
c(t,t_i,t_{i+1}) := \exp\left\{-\int_t^{max(t,t_i)} \sigma_{t_{i+1},t_N}(u)\sigma_{t_i,t_{i+1}}(u)\,du - \int_{max(t,t_i)}^{t_{i+1}} \sigma_{t_{i+1},t_N}(u)\sigma_{S,t_{i+1}}(u)\,du\right\}
$$

Proof. ad (a) Pricing under *No–Arbitrage* implies that at time $t \in [t_0, t_{i+1}]$

$$
\begin{aligned}
C_t^{(a)} &= E_{P^*}\left[e^{-\int_t^{t_N} r_u\,du}\bar\beta_{t_{i+1},t_N}\left(\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right)^+ \middle|\, \mathcal{F}_t\right] \\
&= E_{P^*}\left[e^{-\int_t^{t_{i+1}} r_u\,du}\left(\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right)^+ \middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

This is equal to the t–price of an exchange option with maturity t_{i+1} and payoff $(X_{t_{i+1}}^{(a)} - Y_{t_{i+1}}^{(a)})^+$ where

$$
X_{t_{i+1}}^{(a)} = \frac{S_{t_{i+1}}}{S_{t_i}}\ ,\quad Y_{t_{i+1}}^{(a)} = e^{g(t_{i+1}-t_i)}.
$$

Furthermore, it holds that for $0 \leq t \leq t_{i+1}$

$$\begin{aligned} X_t^{(a)} &= E_{P^*}\left[e^{-\int_t^{t_{i+1}} r_u\,du}\, X_{t_{i+1}}^{(a)} \middle| \mathcal{F}_t \right] \\ &= E_{P^*}\left[e^{-\int_t^{t_i} r_u\,du} E_{P^*}\left[e^{-\int_{t_i}^{t_{i+1}} r_u\,du} \frac{S_{t_{i+1}}}{S_{t_i}} \middle| \mathcal{F}_{t_i} \right] \middle| \mathcal{F}_t \right] 1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}} 1_{\{t\in[t_i,t_{i+1}]\}} \\ Y_t^{(a)} &= E_{P^*}\left[e^{-\int_t^{t_{i+1}} r_u\,du}\, Y_{t_{i+1}}^{(a)} \middle| \mathcal{F}_t \right] = D(t,t_{i+1}) e^{g(t_{i+1}-t_i)}. \end{aligned}$$

The calculation of $X^{(b)}$ and $Y^{(b)}$ is analogous. However, the computation of $X^{(b)}$ is based on Girsanov's theorem. In particular, there is more than one change of measure involved, c.f. appendix A. □

Observe that the price processes of the assets X and Y are lognormal.[3] This implies that there exists a continuous–time hedging strategy which is, if specified in X and Y, uniquely defined. Obviously, X and Y are not necessarily traded assets. The topic of hedging is analyzed in detail in section 6. For the time being, we are purely interested in the fair options prices.[4] Notice, that the pricing formula for exchange options is well known in a Black/Scholes–type setup and was first derived by Margrabe (1978). This result was extended to a more general setup which includes stochastic interest rates by Frey and Sommer (1996). Applying these results gives the following proposition.

Proposition 3.3. *For $t \in [t_0, t_{i+1}]$ and $w \in \{a, b\}$, the t–price of the embedded option C is given by*

$$C_t^{(w)} = X_t^{(w)} \mathcal{N}\left(h_1(t, t_i, t_{i+1}, Z_t^{(w)}) \right) - Y_t^{(w)} \mathcal{N}\left(h_2(t, t_i, t_{i+1}, Z_t^{(w)}) \right)$$

where $Z_t^{(w)} := \frac{X_t^{(w)}}{Y_t^{(w)}}$. $\mathcal{N}$ denotes the cumulative distribution function of the standard normal distribution and the functions h_1 and h_2 are given by

$$\begin{aligned} h_1(s,t,u,z) &= \frac{\ln(z) + \frac{1}{2}v^2(s,t,u)}{v(s,t,u)} \qquad h_2(s,t,u,z) = h_1(s,t,u,z) - v(s,t,u), \\ \textit{where} \quad v^2(s,t,u) &= \int_s^{max(s,t)} \|\sigma_{t,u}(x)\|^2\,dx + \int_{max(s,t)}^u \|\sigma_{S,u}(x)\|^2\,dx. \end{aligned}$$

Proof. In Frey and Sommer (1996) it is shown that, in a model where the quotient process $Z := \frac{X}{Y}$ is lognormal, the price of an option to exchange X for Y at maturity date T with payoff $[X_T - Y_T]^+$ is given by

$$C(t, Z_t) = X_t \mathcal{N}(h_1(t, Z_t)) - Y_t \mathcal{N}(h_2(t, Z_t))$$

---

[3]For a succinct treatment of the significance of this assumption see Rady (1997).

[4]Again, it is to emphasize that prices are only fair with respect to a model assumptions. For example, lifting the assumption that the volatilities and correlation coefficients are known, a price is only as meaningful as the hedging strategy behind it is robust with respect to the misspecification problematic.

where $Z := \frac{X}{Y}$

$$h_1(t,z) = \frac{\ln(z) + \frac{1}{2}\int_t^T \|\sigma_Z(s)\|^2 ds}{\sqrt{\int_t^T \|\sigma_Z(s)\|^2 ds}}, \quad h_2(t,z) = h_1(t,z) - \sqrt{\int_t^T \|\sigma_Z(s)\|^2 ds}.$$

Therefore, one only needs to compute the (total) volatilities of $Z^{(a)}$ and $Z^{(b)}$ from time t to time $T = t_N$. Notice that the quotient process of two lognormal processes is lognormal as well, in particular $\sigma_Z = \sigma_X - \sigma_Y =: \sigma_{X,Y}$. Furthermore, Proposition 3.2

$$\begin{aligned} Z_t^{(a)} &= \frac{X_t^{(a)}}{Y_t^{(a)}} = \frac{D(t,t_i)}{D(t,t_{i+1})e^{g(t_{i+1}-t_i)}} 1_{\{t\in[t_0,t_i[\}} + \frac{\frac{S_t}{S_{t_i}}}{D(t,t_{i+1})e^{g(t_{i+1}-t_i)}} 1_{\{t\in[t_i,t_{i+1}]\}} \\ Z_t^{(b)} &= \frac{X_t^{(b)}}{Y_t^{(b)}} = c(t,t_i,t_{i+1})\frac{X_t^{(a)}}{Y_t^{(a)}} = c(t,t_i,t_{i+1})Z_t^{(a)}. \end{aligned}$$

The total volatility v_{t,t_N} from time t to t_N of the embedded option does not depend on the choice of $\bar{\beta}_{t_{i+1},t_N}$. In particular, we have

$$v_{t,t_N}^{Z^{(a)}} = v_{t,t_N}^{Z^{(b)}} = v_{t,t_{i+1}} := v(t,t_i,t_{i+1})$$

where

$$v^2(t,t_i,t_{i+1}) = \int_t^{max(t,t_i)} \|\sigma_{t_i,t_{i+1}}(s)\|^2 \, ds + \int_{max(t,t_i)}^{t_{i+1}} \|\sigma_{S,t_{i+1}}(s)\|^2 \, ds. \quad \square$$

The above proposition gives the price of the embedded option in terms of the assets X and Y. It is worth mentioning that the embedded option price $C^{(a)}$ is independent of the correlation of asset prices and interests rates between t_{i+1} and t_N. This is easily explained looking at the proof of Proposition 3.2. With respect to the expectation of the discounted payoff, the discount factor of t_{i+1} until t_N and the stochastic bank account cancel out such that the delayed payment option can also be interpreted as an option with maturity t_{i+1}, i.e. an option without a time delay of payments. The asset prices underlying the t–price of the exchange option in scenario (b), i.e. $X_t^{(b)}$ and $Y_t^{(b)}$, are on the one hand both proportional to the asset prices $X_t^{(a)}$ and $Y_t^{(a)}$. The proportionality factor is given by

$$\bar{\beta}_{t_{i+1},t_N}\frac{D(t,t_N)}{D(t,t_{i+1})}.$$

On the other hand, the price of the (pseudo–) asset $X^{(b)}$ is additionally corrected with a parameter c which is determined by forward volatilities. c depends on the correlation of the asset S with a bond maturing at t_N and t_{i+1}.[5] The following corollary anticipates the convexity correction of the option price which is an implication of scenario (b). A detailed discussion and illustration of the effects and implications are given in sec. 5. Applying the pricing results of Proposition 3.3, the convexity correction between the deterministic and stochastic accumulation factor is given by

[5]This is implied by the product of the forward volatility σ_{t_{i+1},t_N} and the asset volatility σ_S.

Corollary 3.4. *Let $C^{(a)}$ denote the embedded option price with stochastic accumulation factor and $C^{(b)}$ denote the embedded option price with a deterministic accumulation factor $\bar{\beta}_{t_{i+1},t_N}$, then it holds*

$$C^{(b)}(t, X_t^{(b)}, Y_t^{(b)}) = \bar{\beta}_{t_{i+1},t_N} \tfrac{D(t,t_N)}{D(t,t_{i+1})} C^{(a)}(t, c(t, t_i, t_{i+1}) X_t^{(a)}, Y_t^{(a)})$$

where c is given as in Proposition 3.2.

4. Fair contract specification

Recall that a contract is called fair if the present value of the periodic premia A coincides with the contract value at the date of contract inception t_0, i.e.

$$A \sum_{i=0}^{N-1} D(t_0, t_i) = D(t_0, t_N)\, \tilde{A}_{t_{N-1}}\, e^{g(t_N - t_{N-1})} + \alpha \sum_{i=0}^{N-1} \tilde{A}_{t_i} C_{t_0}^{(i)}. \tag{7}$$

Since $\tilde{A}$ is proportional to A, the fair contract parameters (α, g) are independent of the size of the periodic premia A. Furthermore, note that the difference between the two present value terms, i.e.

$$A \sum_{i=0}^{N-1} D(t_0, t_i) - D(t_0, t_N)\, \tilde{A}_{t_{N-1}}\, e^{g(t_N - t_{N-1})}$$

can be interpreted as an implicit premium for the option features granted in the contract. According to equation (7), a fair participation rate α^* is given by

$$\alpha^* = \frac{A \sum_{i=0}^{N-1} D(t_0, t_i) - D(t_0, t_N) \tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})}}{\sum_{i=0}^{N-1} \tilde{A}_{t_i} C_{t_0}^{(i)}}.$$

The denominator is positive because of the option features such that α^* is positive if the numerator is positive, too, i.e. if

$$A \sum_{i=0}^{N-1} D(t_0, t_i) - D(t_0, t_N) \tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})} > 0.$$

Intuitively it is clear that the implicit option premium is positive if the guaranteed amount in terms of a rate g is lower than the amount which can be achieved if the present value of the periodic premia is invested on the market today. Therefore, the following argumentation is based on the forward yields which are observed today. Let $y(t_i, t_j, t_k)$ $(i \le j \le k)$ denote the forward yield at t_i which holds for the interval $[t_j, t_k]$, i.e.

$$y(t_0, t_i, t_N) := \frac{-1}{t_N - t_i} \ln \frac{D(t_0, t_N)}{D(t_0, t_i)}.$$

In particular, $y(t_0, t_0, t_i)$ denotes the spot yield. The present value of the difference of the paid premia and the guaranteed amount G can be rewritten in terms of the

Fair parameter combinations (α^*, g^*)

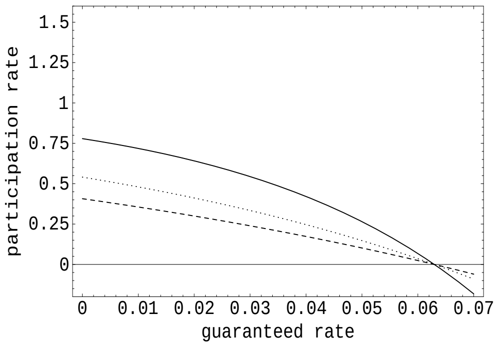

FIGURE 2. Fair parameter combinations for a contract of type (a) and maturity in 30 years for three varying asset volatilities.

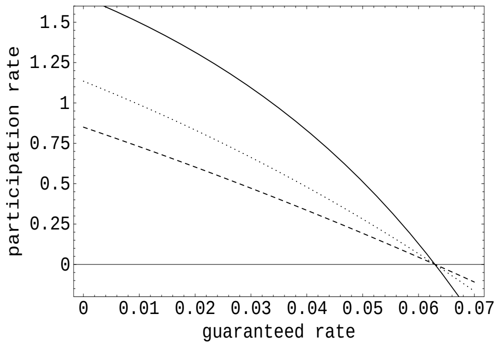

FIGURE 3. Fair parameter combinations for a contract of type (b) with $\bar{\beta} = 1$ and maturity in 30 years for three varying asset volatilities.

The interest rate dynamic is assumed to be as in a Vasicek model with $r_0 = 0.05$, volatility 0.02, speed of mean reversion 0.18 and long run mean 0.07. The asset dynamic is given by a Black/Scholes model with volatility of 0.1, 0.2 and 0.3, respectively. The correlation of assets and bonds is 0.01.

forward yields by

$$
\begin{aligned}
& A \sum_{i=0}^{N-1} D(t_0, t_i) - D(t_0, t_N) \tilde{A}_{t_{N-1}} e^{g(t_N - t_{N-1})} \\
= \quad & D(t_0, t_N) \left(A \sum_{i=0}^{N-1} \frac{D(t_0, t_i)}{D(t_0, t_N)} - A \sum_{i=0}^{N-1} e^{g(t_N - t_i)} \right) \\
= \quad & A \;\; D(t_0, t_N) \sum_{i=0}^{N-1} \left(e^{y(t_0, t_i, t_N)(t_N - t_i)} - e^{g(t_N - t_i)} \right).
\end{aligned}
$$

According to the intuition above, a guaranteed rate g which is lower than the forward yield is a sufficient condition for a positive fair participation rate α^*, i.e.

$$g \leq min_{i=1,\dots,N-1} \; y(t_0, t_i, t_N) \quad \Rightarrow \quad \alpha^* \geq 0.$$

In particular, for a flat yield curve, the condition simplifies to

$$\alpha^* \geq 0 \Longleftrightarrow g \leq y.$$

A second question which is of minor importance is if the fair parameter α^* is bounded from above by one. Notice that in the case of a deterministic factor $\bar{\beta}$, the answer simply is no, i.e. α^* is not bounded from above by one. This is of course due to the loss which is incurred by a long time delay in combination with a low factor $\bar{\beta}$. In particular, α^* may be greater than one if no interest is granted for the time delay of the payments. This is illustrated in Figure 3.

Under scenario (a), i.e. $\bar{\beta}$ coincides with the stochastic bank account, the property $\alpha^* \leq 1$ is equivalent to the condition that the sum of weighted option prices is greater than the compensation to be granted if the guaranteed rate g is lower than the forward rates.

$$\alpha^* \leq 1 \iff \sum_{i=0}^{N-1} \tilde{A}_{t_i} C_{t_0}^{(i)} \geq A \;\; D(t_0, t_N) \sum_{i=0}^{N-1} \left(e^{y(t_0,t_i,t_N)(t_N - t_i)} - e^{g(t_N - t_i)} \right).$$

Notice that the fair contract parameter α^* is decreasing in g. Therefore, it is enough to consider the limit $g \to -\infty$. We have

$$\begin{aligned} \lim_{g \to -\infty} A \;\; D(t_0, t_N) \sum_{i=0}^{N-1} \left(e^{y(t_0,t_i,t_N)(t_N - t_i)} - e^{g(t_N - t_i)} \right) &= A \;\; D(t_0, t_N) \sum_{i=0}^{N-1} e^{y(t_0,t_i,t_N)(t_N - t_i)} \\ &= A \sum_{i=0}^{N-1} D(t_0, t_i) \end{aligned}$$

and

$$\begin{aligned} \lim_{g \to -\infty} \sum_{i=0}^{N-1} \tilde{A}_{t_i} C_{t_0}^{(i)} &= \lim_{g \to -\infty} \sum_{i=0}^{N-1} A \left(\sum_{j=0}^{i} e^{g(t_i - t_j)} \right) C_{t_0}^{(i)} \\ &= \lim_{g \to -\infty} \sum_{i=0}^{N-1} A \left(\sum_{j=0}^{i-1} e^{g(t_i - t_j)} + 1 \right) C_{t_0}^{(i)} \\ &= A \sum_{i=0}^{N-1} D(t_0, t_i) \end{aligned}$$

such that the limits coincide. Thus, in the case of scenario (a), $\lim_{g \to -\infty} \alpha^*(g) = 1$ and α^* is bounded from above by one. This is illustrated in Figure 2.

5. Certainty equivalent

It turned out that in the case of scenario (a) the delayed payment option can be reinterpreted as a simple option, i.e. an option without a time lag between the observation of the excess return and the final payment. However, this is not possible if the accumulation factor is deterministic. To measure the difference between these two cases consider a deterministic accumulation factor such that the price of deferring the payment is zero. More precisely, we call $\bar{\beta}^*$ the *certainty equivalent* if the value of the delayed payment option with the deterministic accumulation factor $\bar{\beta}^*$ coincides with the value of the option without time delay. This means that a deterministic $\bar{\beta}$ which is higher (lower) than $\bar{\beta}^*$ implies a positive (negative) premium for the time lag of payoffs. For $\bar{\beta} = \bar{\beta}^*$ the time lag of the payoffs is fully compensated. Applying Corollary 3.4 the certainty equivalent $\bar{\beta}^*$ is given by

$$\bar{\beta}^*_{t_{i+1}, t_N} := \frac{D(t_0, t_{i+1}) \, C^{(a)}(t_0, X_{t_0}^{(a)}, Y_{t_0}^{(a)})}{D(t_0, t_N) \, C^{(a)}(t_0, c(t_0, t_i, t_{i+1}) X_{t_0}^{(a)}, Y_{t_0}^{(a)})}$$

where the parameter c is given as in Proposition 3.2.

Notice that the certainty equivalent is connected to the t_N–forward price of a zero coupon bond with maturity t_{i+1}. Intuitively it is clear that in a world without stochastic interest rates the certainty equivalent coincides with the forward price, i.e.

Certainty equivalent

ρ	$\frac{1}{t_N - t_{i+1}} \ln \bar{\beta}^*(t_{i+1}, t_N)$	$\bar{\beta}^*(t_{i+1}, t_N)$
-1.00	0.0720282	1.91220
-0.90	0.0708124	1.89139
-0.80	0.0695942	1.87077
-0.70	0.0683737	1.85033
-0.60	0.0671509	1.83008
-0.50	0.0659256	1.81001
-0.40	0.0646979	1.79012
-0.30	0.0634678	1.77041
-0.20	0.0622352	1.75088
-0.10	0.0610001	1.73152
0.00	0.0597625	1.71234
0.10	0.0585224	1.69334
0.20	0.0572796	1.67450
0.30	0.0560343	1.65584
0.40	0.0547863	1.63735
0.50	0.0535357	1.61902
0.60	0.0522824	1.60086
0.70	0.0510264	1.58287
0.80	0.0497677	1.56504
0.90	0.0485062	1.54737
1.00	0.0472419	1.52986

TABLE 1. Certainty equivalent with respect to an embedded option where $t = t_i = 0$, $t_N = 10$ and $g = 0.03$. In particular, notice that the certainty equivalent is slightly above the forward price $\frac{D(t,t_{i+1})}{D(t,t_N)} = 1.70653$ for $\rho = 0$.

$c = 1$.[6] Besides, recalling that the option price C is increasing in the price of the underlying X it is easy to summarize the connection of the certainty equivalent and the convexity correction parameter c and its implications as follows.

LEMMA 5.1. *Let $F(t_0, t_{i+1}, t_N)$ denote the t_N–forward price of a zero coupon bond with maturity t_{i+1}, i.e.*

$$F(t_0, t_{i+1}, t_N) := \frac{D(t_0, t_{i+1})}{D(t_0, t_N)}.$$

The certainty equivalent $\bar{\beta}^*_{t_{i+1},t_N}$ is above (below) $F(t_0, t_{i+1}, t_N)$ if and only if $c = c(t_0, t_i, t_{i+1})$ is below (above) one, equality holds for $c = 1$.

The convexity correction parameter as well as the certainty equivalent are crucially depending on the correlation of assets and bonds. Heuristically, this is easily explained along the lines of the motivation which is already given in the introduction and in Section 3. With respect to scenario (a), the option price given by

$$E_{P^*}\left[e^{-\int_{t_0}^{t_{i+1}} r_u \, du} \left(\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)} \right)^+ \right]$$

[6] It is easily seen that $c = 1$ in the limit for interest (or bond) volatilities converging to zero.

is equal to an option with maturity t_{i+1} and no time delay. On the other hand, with respect to scenario (b), the price of the delayed payment option, i.e.

$$\bar{\beta}_{t_{i+1},t_N} E_{P^*}\left[e^{-\int_{t_0}^{t_{i+1}} r_u\,du} e^{-\int_{t_{i+1}}^{t_N} r_u\,du}\left(\frac{S_{t_{i+1}}}{S_{t_i}} - e^{g(t_{i+1}-t_i)}\right)^+\right]$$

is increasing in the correlation of the asset and the bond with maturity t_{i+1} and that of the asset and the t_N–bond. Therefore, it is clear that the certainty equivalent is decreasing in the asset bond correlation. In order to formalize the above observations, let $\rho_{S,\bar{t}}$ denote the local correlation coefficient of the asset S and a bond with maturity $\bar{t}$, i.e.

$$\rho_{S,\bar{t}}(t) := \frac{d\langle S, D(.,\bar{t})\rangle_t}{\sqrt{d\langle S\rangle_t\, d\langle D(.,\bar{t})\rangle_t}}\,.$$

If the dynamic of S is given by equation (5) and the dynamic of D by equation (6) we have

$$\rho_{S,\bar{t}}(t) = \frac{\sigma_s(t)\sigma_{\bar{t}}(t)}{\sqrt{\|\sigma_S(t)\|^2\|\sigma_{\bar{t}}(t)\|^2}} \quad \text{respectively} \quad \sigma_s(t)\sigma_{\bar{t}}(t) = \rho_{S,\bar{t}}(t)\|\sigma_S(t)\|\|\sigma_{\bar{t}}(t)\|.$$

Together with the definition of the convexity correction, c.f. Proposition 3.2, c can be written in terms of the asset bond correlation, i.e.

$$\begin{aligned} c(t_0,t_i,t_{i+1}) &= \exp\left\{\int_{t_0}^{t_i} \sigma_{t_N,t_{i+1}}(u)\sigma_{t_i}(u)\,du - \int_{t_0}^{t_{i+1}} \sigma_{t_N,t_{i+1}}(u)\sigma_{t_{i+1}}(u)\,du\right\} \\ &\quad \exp\left\{\int_{t_i}^{t_{i+1}} \|\sigma_S(u)\|\left(\rho_{S,t_N}(u)\|\sigma_{t_N}(u)\| - \rho_{S,t_{i+1}}(u)\|\sigma_{t_{i+1}}(u)\|\,du\right)\right\}. \end{aligned}$$

Thus, depending on the variance–covariance matrix of assets and bonds, the certainty equivalent can be above, below or less than the forward price F. In particular, it is above the forward price if assets and bonds are uncorrelated. This is illustrated by the Vasicek model where the parameters are given as in Figure 2 and Figure 3, respectively.

6. Hedging

In the following section we analyze the hedging decisions needed for the payoff of the insurance contract to be synthesized on the financial market. We will show that the hedging of the contract type (a) is much easier in its construction and has also the advantage that it can be specified without using the asset bond correlation as a separate input. It turns out that this is not only appealing with respect to the problem of parameter estimation, but also with regard to the so–called hedging robustness.

Again, we simplify notation and use C instead of $C^{(i)}$ to denote the delayed payment option with reference period $[t_i, t_{i+1}]$ and payoff at t_N. As hedging is trivial once the excess return is known, we focus on the hedging decisions for $t \in [t_0, t_{i+1}]$. Along the lines of Proposition 3.3 we have

Proposition 6.1. *For $t \in [t_0, t_{i+1}]$ and $w \in \{a, b\}$, the self–financing and duplicating stratgey $\phi = \left(\phi^{X^{(w)}}, \phi^{Y^{(w)}}\right)$ in the assets $X^{(w)}, Y^{(w)}$ for the embedded option C is* *given by*

$$\phi_t^{X^{(w)}} = \mathcal{N}\left(h_1(t, t_i, t_{i+1}, Z_t^{(w)})\right), \quad \phi_t^{Y^{(w)}} = -\mathcal{N}\left(h_2(t, t_i, t_{i+1}, Z_t^{(w)})\right)$$

where $Z_t^{(w)} := \frac{X_t^{(w)}}{Y_t^{(w)}}$. $\mathcal{N}$ *denotes the cumulative distribution function of the standard normal distribution and the functions* $h^{(1)}$ *and* $h^{(2)}$ *are given by*

$$h_1(s,t,u,z) \;:=\; \frac{\ln(z) + \frac{1}{2}v^2(s,t,u)}{v(s,t,u)}; \quad h_2(s,t,u,z) := h_1(s,t,u,z) - v(s,t,u),$$

$$\text{where} \quad v^2(s,t,u) \;:=\; \int_s^{max(s,t)} \|\sigma_{t,u}(x)\|^2\,dx + \int_{max(s,t)}^u \|\sigma_{S,u}(x)\|^2\,dx.$$

Proof. In a model where the quotient process $Z := \frac{X}{Y}$ is lognormal, it is well known, c.f. for example Margrabe (1978) or Frey and Sommer (1996), that the hedging strategy of an option to exchange X for Y at maturity date T with payoff $[X_T - Y_T]^+$ is given by

$$\phi_t^X = \mathcal{N}(h^{(1)}(t, Z_t)) \text{ units of } X, \quad \phi_t^Y = -\mathcal{N}(h^{(2)}(t, Z_t)) \text{ units of } Y,$$

where $Z := \frac{X}{Y}$. The rest of the proof is a direct consequence of Proposition 3.3.

□

Observing that the assets $X^{(w)}, Y^{(w)}$ are not traded on the financial market in their original version, a natural way to proceed is to synthesize $X^{(w)}, Y^{(w)}$ with basic assets. This is easy in scenario (a). Recall that

$$\begin{aligned} X_t^{(a)} &= D(t,t_i)1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}}1_{\{t\in[t_i,t_{i+1}]\}}, \\ Y_t^{(a)} &= D(t,t_{i+1})e^{g(t_{i+1}-t_i)}1_{\{t\in[t_0,t_{i+1}]\}}. \end{aligned}$$

Thus, it is immediately clear that $X^{(a)}$ is created by the following strategy. For $t \in [t_0, t_i[$ buy a bond with maturity t_i and sell the bond at $t = t_i$. At t_i, the portfolio value is $D(t_i, t_i) = 1$ which is exactly the amount needed to buy $\frac{1}{S_{t_i}}$ assets S. In particular, the above strategy which consists of two buy and hold decisions is self–financing and duplicates the asset $X^{(a)}$. Besides, buying $e^{g(t_{i+1}-t_i)}$ bonds with maturity t_{i+1} synthesizes $Y^{(a)}$. Thus, the following proposition is straightforward.

Proposition 6.2. *The strategy* $\tilde{\phi} = \left(\tilde{\phi}^{(i)}, \tilde{\phi}^{(i+1)}, \tilde{\phi}^S\right)$ *consisting of the assets* $D(.,t_i), D(.,t_{i+1})$ *and* S *where*

$$\begin{aligned} \tilde{\phi}_t^{(i)} &= \mathcal{N}\left(h_1(t,t_i,t_{i+1},Z_t^{(a)})\right) 1_{\{t\in[t_0,t_i[\}} \\ \tilde{\phi}_t^{(i+1)} &= -e^{g(t_{i+1}-t_i)}\mathcal{N}\left(h_2(t,t_i,t_{i+1},Z_t^{(a)})\right) \\ \tilde{\phi}_t^S &= \mathcal{N}\left(h_1(t,t_i,t_{i+1},Z_t^{(a)})\right) 1_{\{t\in[t_i,t_{i+1}]\}} \end{aligned}$$

gives a perfect hedge for the delayed payment option C *regarding scenario (a).*

There are two things worth mentioning. In the first instance, the only unobservable parameter of the hedging strategy is the cumulated volatility v. In particular, this implies that the estimation problem does not include a separate estimation of the covariances. Secondly, notice that the strategy, apart from indicating a switch from the hedging instrument $D(.,t_i)$ to S, is a simple Black/Scholes–type strategy. This allows the application of the well known robustness result of Black/Scholes, c.f. El Karoui, Jeanblanc-Picqué and Shreve (1998), which states that the associated cost process is of finite variation irrespective of the true dynamics of the underlying assets.[7] If an upper bound for all local volatilities of the quotient process $Z^{(a)}$ is known, the hedging strategy which is obtained for the maximum volatility superreplicates the option, i.e. it dominates the payoff to be hedged almost surely. In particular, a conservative estimation of v gives an upper price bound.

Unfortunately, this is not true with respect to a delayed payment option with respect to scenario (b). Recall that the price process of the asset $X^{(b)}$ is given by

$$X_t^{(b)} = \bar{\beta}_{t_{i+1},t_N} \frac{D(t,t_N)}{D(t,t_{i+1})} c(t,t_i,t_{i+1}) \left(D(t,t_i) 1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}} 1_{\{t\in[t_i,t_{i+1}]\}} \right).$$

In the Appendix B it is shown that one (but not the only) possibility to synthesize $X^{(b)}$ is given by a strategy $\phi = \left(\phi^{(i)}, \phi^{(i+1)}, \phi^{(N)}, \phi^{S}\right)$ in the basic assets $D(.,t_i), D(.,t_{i+1}), D(.,t_N)$ and S where

$$\begin{aligned} \phi_t^{(N)} &= \frac{X_t^{(b)}}{D(t,t_N)}, \quad \phi_t^{(i+1)} = \frac{-X_t^{(b)}}{D(t,t_{i+1})}, \\ \phi_t^{(i)} &= \frac{X_t^{(b)}}{D(t,t_i)} 1_{\{t\in\{t_0,t_i\}\}}, \quad \phi_t^{S} = \frac{X_t^{(b)}}{S_t} 1_{\{t\in\{t_i,t_{i+1}\}\}}. \end{aligned}$$

Notice that the above strategy depends on the convexity correction parameter c. In particular, this is true for all strategies which can be used to synthesize $X^{(b)}$ without using $X^{(b)}$ itself. The above strategy is chosen as the most suitable one in the following sense. Once the asset $X^{(b)}$, i.e. the correction parameter c, is known, the strategy no longer depends on the model assumption. However, c depends on the asset bond correlation which must be estimated to construct the hedge. Besides, it is not possible to hedge the insurance contract with a robust Black/Scholes type strategy regarding scenario (b). Basically, this is explained as follows. The process $X^{(b)}$ is not the *true* underlying in a world where the volatilities are stochastic themselves. Therefore, the hedging error is not only due to a potentially misspecified volatility, but also to the wrong hedging instrument. The last source gives rise to costs which are not of finite variation, which is a necessary condition to achieve a superhedge, c.f. Dudenhausen, Schlögl and Schlögl (1998). In particular, the knowledge of an upper volatility bound is not enough to construct a robust hedge or give a meaningful upper price bound.

7. Conclusion

The results of the paper show that the fairness and the hedging possibilities of an insurance contract where excess returns are observed periodically but delayed to maturity, depend on the question of how the time–lag is honored in terms of an accumulation factor.

It turns out that the choice of a stochastic accumulation factor facilitates the pricing and risk management with respect to the following aspects. Firstly, the parameter

[7]The true process is assumed to be a diffusion process.

estimation problem is reduced. It is not necessary to estimate the correlations of the asset and the bonds separately. Secondly, the hedging strategy implied by a Black/Scholes type model setup is robust, i.e. if the strategy is computed according to an upper bound on the (forward–) volatilities, it dominates the payoff to be hedged. Once a conservative estimation, i.e. a confidence level, of the volatilities is achieved, it is ensured that the insurance company *stays on the safe side*. This also means that the guarantees which are granted to the insurance takers can be honored with probability one if the volatility does not violate the upper bound. However, this is not true for a deterministic accumulation factor, as is common for many unit–linked contracts e.g. in Germany. At first glance, the use of a deterministic accumulation factor might be motivated to reduce the contract complexity for the underwriter. Our results show that this is not true. At least three arguments should be mentioned.

Firstly, pricing is affected by the convexity correction which is not only proportional to the forward price, but also sensitive to changes in the bond asset correlation. Thus, both a shift of the yield curve and a change in the correlation structure have an impact on the fair contract specification. Bearing in mind that in contrast to pure financial products the pricing of unit–linked (life) insurance products is not at all adjusted continuously, the dimension of this problem becomes obvious.

Secondly, since pricing is much more static for insurance products than for pure financial products, hedging needs closer attention. One problem of hedging is the long time to maturity of the contracts. This is true for both scenarios. But even if a static or quasi–static hedge is available in the case of a stochastic accumulation factor, this hedge is not possible for the opposite case. For example, hedging long time to maturity with short time to maturity options can be the basis of a superhedging strategy in the first, but not in the second case. That is, a deterministic accumulation factor implies a higher necessity of dynamic hedging.

Thirdly, consider risk management. To some extent, pricing can be seen as a mathematical problem and continuous–time hedging as a theoretical concept. However, the risk management affects the default of the underwriter as well as the benefits to the insured. Therefore, the risk management concerns the insurer and the stock holder as well as the regulator. One way to overcome pricing and hedging problems is to set up a financial strategy which dominates financial obligations. Robustness against model misspecifications of such superhedging strategies can be seen as a necessary requirement. In the case of a deterministic accumulation factor, it is not possible to hedge the financial claim with a robust Black/Scholes–type strategy. In this case, fixing an upper bound for the volatility does not necessarily specify the worst–case scenario. Therefore, a Black/Scholes–type model is not sufficient to construct a robust hedge or to calculate meaningful upper price bounds. In order to formulate and implement a risk management system covering the needs of the insurer and the regulation, it is far from sufficient just to calculate some kind of value at risk or related measure. To sum up our discussion, the choice of the accumulation factor in unit–linked insurance contracts plays a critical role and its construction should be reconsidered.

Appendix A. Proof of Proposition 3.2

Lemma A.1. *Let $W^{t_{i+1}}$ denote a d–dimensional Brownian Motion with respect to the t_{i+1}–forward measure.*

(a) *For $t \in [t_0, t_i]$ it holds*

$$\begin{aligned}\frac{S_{t_{i+1}}}{S_{t_i}} = \frac{D(t,t_i)}{D(t,t_{i+1})} \quad & \exp\left\{-\frac{1}{2}\int_t^{t_i} \|\sigma_{t_i,t_{i+1}}(u)\|^2\,du + \int_t^{t_i} \sigma_{t_i,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\} \\ \cdot \quad & \exp\left\{-\frac{1}{2}\int_{t_i}^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_{t_i}^{t_{i+1}} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\}\end{aligned}$$

(b) *For $t \in [t_i, t_{i+1}]$ it holds*

$$\frac{S_{t_{i+1}}}{S_t} = \frac{1}{D(t,t_{i+1})}\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_{i+1}} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\}$$

Proof. ad (a) With respect to the t_{i+1}–forward measure it holds for $t \le t_i$

$$\frac{S_{t_{i+1}}}{D(t_{i+1},t_{i+1})} = \frac{S_t}{D(t,t_{i+1})}\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_{i+1}} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\},$$

with respect to the t_i–forward measure

$$\frac{S_{t_i}}{D(t_i,t_i)} = \frac{S_t}{D(t,t_i)}\exp\left\{-\frac{1}{2}\int_t^{t_i} \|\sigma_{S,t_i}(u)\|^2\,du + \int_t^{t_i} \sigma_{S,t_i}(u)\,dW^{t_i}(u)\right\}.$$

Therefore,

$$\frac{S_{t_{i+1}}}{S_{t_i}} = \frac{D(t,t_i)}{D(t,t_{i+1})}\frac{\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}\|^2\,dW^{t_{i+1}}(u)\right\}}{\exp\left\{-\frac{1}{2}\int_t^{t_i} \|\sigma_{S,t_i}(u)\|^2\,du + \int_t^{t_i} \sigma_{S,t_i}\,dW^{t_i}(u)\right\}}$$

Using

$$W^{t_i}(t) = W^{t_{i+1}}(t) - \int_0^t \sigma_{t_i,t_{i+1}}(u)\,du$$

$$\begin{aligned}\frac{S_{t_{i+1}}}{S_{t_i}} &= \frac{D(t,t_i)}{D(t,t_{i+1})}\frac{e^{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_{i+1}} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)}}{e^{-\frac{1}{2}\int_t^{t_i} \|\sigma_{S,t_i}(u)\|^2\,du + \int_t^{t_i} \sigma_{S,t_i}(u)\,dW^{t_{i+1}}(u) - \int_t^{t_i} \sigma_{S,t_i}(u)\sigma_{t_i,t_{i+1}}(u)\,du}} \\ &= \frac{D(t,t_i)}{D(t,t_{i+1})}\exp\left\{-\frac{1}{2}\int_t^{t_i} \|\sigma_{S,t_{i+1}}(u)\|^2 - \|\sigma_{S,t_i}(u)\|^2 - 2\sigma_{S,t_i}(u)\sigma_{t_i,t_{i+1}}(u)\,du\right\} \\ &\quad \exp\left\{\int_t^{t_i} \left(\sigma_{S,t_{i+1}}(u) - \sigma_{S,t_i}(u)\right)\,dW^{t_{i+1}}(u)\right\} \\ &\quad \exp\left\{-\frac{1}{2}\int_{t_i}^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_{t_i}^{t_{i+1}} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\}\end{aligned}$$

Using $\sigma_{S,t_{i+1}} - \sigma_{S,t_i} = \sigma_{t_i,t_{i+1}}$ and $\sigma_{S,t_{i+1}} = \sigma_{S,t_i} + \sigma_{t_i,t_{i+1}}$ gives the result.
ad (b). (b) is a direct consequence of

$$\frac{S_{t_{i+1}}}{D(t_{i+1},t_{i+1})} = \frac{S_t}{D(t,t_{i+1})}\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,dW^{t_{i+1}}(u)\right\}$$

$\square$.

Lemma A.2. *For* $(X_t)_{0\le t\le t_{i+1}}$ *with*

$$X_t \;:=\; E_{P^*}\left[e^{-\int_t^{t_{i+1}} r_u\,du}\, D(t_{i+1},t_N)\frac{S_{t_{i+1}}}{S_{t_i}}\middle|\,\mathcal{F}_t\right]$$

it holds

$$X_t \;=\; c(t,t_i,t_{i+1})\left[D(t,t_N)\frac{D(t,t_i)}{D(t,t_{i+1})}\mathit{1}_{\{t\in[t_0,t_i[\}} + \frac{D(t,t_N)}{D(t,t_{i+1})}\frac{S_t}{S_{t_i}}\mathit{1}_{\{t\in[t_i,t_{i+1}]\}}\right]$$

where

$$c(t,t_i,t_{i+1}) = \exp\left\{-\int_t^{max(t,t_i)} \sigma_{t_{i+1},t_N}(u)\sigma_{t_i,t_{i+1}}(u)\,du - \int_{max(t,t_i)}^{t_{i+1}} \sigma_{t_{i+1},t_N}(u)\sigma_{S,t_{i+1}}(u)\,du\right\}$$

Proof. With a change of measure, i.e.

$$\left(\frac{dP^*}{dP^{t_{i+1}}}\right)_t = \frac{e^{\int_{t_0}^t r(u)\,du}D(t_0,t_{i+1})}{D(t,t_{i+1})}$$

it follows

$$X_t = D(t,t_{i+1})E_{P^{t_{i+1}}}\left[\frac{D(t_{i+1},t_N)}{D(t_{i+1},t_{i+1})}\frac{S_{t_{i+1}}}{S_{t_i}}\middle|\,\mathcal{F}_t\right]$$

Using

$$\frac{D(t_{i+1},t_N)}{D(t_{i+1},t_{i+1})} = \frac{D(t,t_N)}{D(t,t_{i+1})}\;\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{t_N,t_{i+1}(u)}\|^2\,du + \int_t^{t_{i+1}} \sigma_{t_N,t_{i+1}(u)}\,dW^{t_{i+1}}(u)\right\}$$

$$X_t = D(t,t_N)E_{P^{t_{i+1}}}\left[\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}} \|\sigma_{t_N,t_{i+1}(u)}\|^2\,du + \int_t^{t_{i+1}} \sigma_{t_N,t_{i+1}(u)}\,dW^{t_{i+1}}(u)\right\}\frac{S_{t_{i+1}}}{S_{t_i}}\middle|\,\mathcal{F}_t\right]$$

According to Lemma A.1 it holds for $t\le t_i$

$$\begin{aligned}\frac{S_{t_{i+1}}}{S_{t_i}} = \frac{D(t,t_i)}{D(t,t_{i+1})} \qquad & \exp\left\{-\frac{1}{2}\int_t^{t_i} \|\sigma_{t_i,t_{i+1}}(u)\|^2\,du + \int_{t_i}^{t_{i+1}} \sigma_{t_i,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\}\\ \cdot \quad & \exp\left\{-\frac{1}{2}\int_{t_i}^{t_{i+1}} \|\sigma_{S,t_{i+1}}(u)\|^2\,du + \int_t^{t_i} \sigma_{S,t_{i+1}}(u)\,dW^{t_{i+1}}(u)\right\}\end{aligned}$$

Therefore, for $t\le t_i$

$$\begin{aligned}X_t \;=\;& D(t,t_N)\frac{D(t,t_i)}{D(t,t_{i+1})}\\ & \exp\left\{-\frac{1}{2}\int_t^{t_i}\left(\|\sigma_{t_N,t_{i+1}(u)}\|^2 + \|\sigma_{t_i,t_{i+1}(u)}\|^2\right)\,du\right.\\ & \qquad \left.-\frac{1}{2}\int_{t_i}^{t_{i+1}}\left(\|\sigma_{t_N,t_{i+1}(u)}\|^2 + \|\sigma_{S,t_{i+1}(u)}\|^2\right)\,du\right\}\\ & E_{P^{t_{i+1}}}\left[\exp\left\{\int_t^{t_i}\left(\sigma_{t_N,t_{i+1}(u)}+\sigma_{t_i,t_{i+1}(u)}\right)\,dW^{t_{i+1}}(u)\right.\right.\\ & \qquad\qquad \left.\left.+\int_{t_i}^{t_{i+1}}\left(\sigma_{t_N,t_{i+1}(u)}+\sigma_{S,t_{i+1}(u)}\right)\,dW^{t_{i+1}}(u)\right\}\middle|\,\mathcal{F}_t\right]\\ =\;& D(t,t_N)\frac{D(t,t_i)}{D(t,t_{i+1})}\exp\left\{\int_t^{t_i}\sigma_{t_N,t_{i+1}(u)}\sigma_{t_i,t_{i+1}(u)}\,du + \int_{t_i}^{t_{i+1}}\sigma_{t_N,t_{i+1}(u)}\sigma_{S,t_{i+1}(u)}\,du\right\}\end{aligned}$$

For $t \in [t_i, t_{i+1}]$ it holds (c.f. Lemma A.1 (b)

$$\begin{aligned} X_t &= \frac{D(t,t_N)}{D(t,t_{i+1})}\frac{S_t}{S_{t_i}}\exp\left\{-\frac{1}{2}\int_t^{t_{i+1}}\left(\|\sigma_{t_N,t_{i+1}}(u)\|^2+\|\sigma_{S,t_{i+1}}(u)\|^2\right)du\right\} \\ &\qquad E_{P^{t_{i+1}}}\left[\exp\left\{\int_t^{t_{i+1}}\left(\sigma_{t_N,t_{i+1}}(u)+\sigma_{S,t_{i+1}}(u)\right)dW_{t_{i+1}}(u)\right\}\bigg|\,\mathcal{F}_t\right] \\ &= \frac{D(t,t_N)}{D(t,t_{i+1})}\frac{S_t}{S_{t_i}}\exp\left\{\int_t^{t_{i+1}}\sigma_{t_N,t_{i+1}(u)\sigma_{S,t_{i+1}}(u)\,du}\right\} \end{aligned}$$

$\square$.

Appendix B. Synthesising the pseudo asset $X^{(b)}$

We study the case where the hedge instrument X is not liquidly traded in the market and a potential hedger must use other assets $Y^1, ..., Y^n$ to synthesize X. We place ourselves in a diffusion setting, i.e. the prices X, $Y^1, \ldots, Y^n$ are given by Itô processes which are driven by a d-dimensional Brownian motion W defined on :

$$dX_t = X_t\{\mu_t^X dt + \sigma_t^X dW_t\}$$
$$dY_t^i = Y_t^i\{\mu_t^i dt + \sigma_t^i dW_t\}$$

where μ^X, σ^X and μ^i, σ^i are suitably integrable stochastic processes. We assume the prices X, $Y^1, \ldots, Y^n$ are arbitrage-free. This implies that there is a "market price of risk" process φ such that for any $i \in \{1, \ldots, n\}$:

$$\mu^X - \sigma^X\varphi = \mu^i - \sigma^i\varphi.$$

Synthesizing X out of $Y^1, \ldots, Y^n$ involves finding a self-financing strategy ϕ with a position of ϕ^i in asset Y^i for each $i \in \{1, \ldots, n\}$ such that $X = \sum_{i=1}^n \phi^i Y^i$. The following proposition characterizes these strategies ϕ.

Proposition B.1. *Suppose that $\lambda^1, \ldots, \lambda^n$ are predictable processes satisfying the following two conditions:*

$$(1)\quad \sum_{i=1}^n \lambda_t^i = 1 \qquad (2)\quad \sum_{i=1}^n \lambda_t^i\sigma_t^i = \sigma_t^X.$$

For each $i \in \{1, \ldots, n\}$, we set $\phi^i := \frac{X}{Y^i}\lambda^i$. Then ϕ is a self-financing strategy which identically duplicates X. In particular, any such strategy is of the form above.

Proof: Suppose that weights $\lambda^1, \ldots, \lambda^n$ are given which satisfy conditions (1) and (2) and that ϕ is the corresponding strategy. By condition (1), it is clear that $\sum_{i=1}^n \phi^i Y^i = X$. By the no-arbitrage condition and because of (2) we have

$$\sum_{i=1}^n \lambda^i\mu^i = \sum_{i=1}^n \lambda^i\{\mu^X + \varphi(\sigma^i - \sigma^X)\} = \mu^X.$$

From this we see that ϕ is also self-financing because

$$\sum_{i=1}^n \phi_t^i dY_t^i = X_t\sum_{i=1}^n \lambda_t^i\{\mu_t^i dt + \sigma_t^i dW_t\} = X_t\{\mu^X dt + \sigma_t^X dW_t\} = dX_t.$$

Conversely, if ϕ is a self-financing strategy which identically duplicates X, then the weights $\lambda^1, \ldots, \lambda^n$ determined by $\lambda^i := \frac{Y^i}{X}\phi^i$ will satisfy the two conditions.

□

The weights $\lambda^1, \ldots, \lambda^n$ are to be interpreted as portfolio weights, i.e. λ^i is the proportion of total capital to be invested in asset Y^i.

A question that arises naturally is whether a duplication strategy exists. Irrespective of the concrete choice of X, this is only true if the market determined by $Y^1, \ldots, Y^n$ is dynamically complete. However, the proposition below deals with the concrete application of how to synthesize the pseudo asset X of the insurance scenario (b), i.e.

Proposition B.2. *The asset X with price process*

$$X_t = \bar{\beta}_{t_{i+1},t_N} \frac{D(t,t_N)}{D(t,t_{i+1})} c(t,t_i,t_{i+1}) \left(D(t,t_i) 1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}} 1_{\{t\in[t_i,t_{i+1}]\}} \right)$$

is a redundant asset in a market where the assets S and the zero coupon bonds with the maturities t_i, t_{i+1} and t_N are traded, i.e. the duplication strategy $\phi = \left(\phi^{(i)}, \phi^{(i+1)}, \phi^{(N)}, \phi^{(S)}\right)$ is given by

$$\begin{aligned} \phi_t^{(N)} &= \frac{X_t}{D(t,t_N)}, \quad \phi_t^{(i+1)} = \frac{-X_t}{D(t,t_{i+1})}, \\ \phi_t^{(i)} &= \frac{X_t}{D(t,t_i)} 1_{\{t\in\{t_0,t_i\}\}}, \quad \phi_t^{(S)} = \frac{X_t}{S_t} 1_{\{t\in\{t_i,t_{i+1}\}\}}. \end{aligned}$$

In particular, the portfolio weights

$$\lambda = \left(1, -1, 1_{\{t\in\{t_0,t_i\}\}}, 1_{\{t\in\{t_i,t_{i+1}\}\}}\right)$$

are independent of the volatility structure.

Proof: Let $(X)^M$ denote the martingale part of the Doob Meyer decomposition of X.

$$\begin{aligned} d\left(X_t^{(i)}\right)^M &= \bar{\beta}_{t_{i+1},t_N} c(t,t_i,t_{i+1})\, d\left[\frac{D(t,t_N)}{D(t,t_{i+1})} \left(D(t,t_i) 1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}} 1_{\{t\in[t_i,t_{i+1}]\}} \right)\right]^M \\ &= \bar{\beta}_{t_{i+1},t_N} c(t,t_i,t_{i+1}) \left[\left(D(t,t_i) 1_{\{t\in[t_0,t_i[\}} + \frac{S_t}{S_{t_i}} 1_{\{t\in[t_i,t_{i+1}]\}} \right) d\left(\frac{D(t,t_N)}{D(t,t_{i+1})}\right)^M \right. \\ &\quad \left. \frac{D(t,t_N)}{D(t,t_{i+1})} \left(dD(t,t_i)^M 1_{\{t\in[t_0,t_i[\}} + d\left(\frac{S_t}{S_{t_i}}\right)^M 1_{\{t\in[t_i,t_{i+1}]\}} \right)\right] \\ &= X_t^{(i)} \left[\sigma_{t_N,t_{i+1}}(t) + \sigma_{t_i}(t) 1_{\{t\in[t_0,t_i[\}} + \sigma_S(t) 1_{\{t\in[t_i,t_{i+1}]\}}\right]. \end{aligned}$$

The rest of the proof follows with Proposition B.1.

□

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